DAVIES-HARRELL REPRESENTATIONS, OTELBAEV'S INEQUALITIES AND PROPERTIES OF SOLUTIONS OF RICCATI EQUATIONS

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ABSTRACT. We consider an equation

$$y''(x)=q(x)y(x),\quad x\in R \tag{1}$$

under the following assumptions on $q(x)$:

$$0\le q(x)\in L_1^{\rm loc}(R),\quad \int_{-\infty}^x q(t)dt>0,\quad \int_x^\infty q(t)dt>0\quad \text{for all}\quad x\in R. \tag{2}$$

Let $v(x)$ (resp. $u(x)$) be a positive non-decreasing (resp. non-increasing) solution of (1) such that

$$v'(x)u(x)-u'(x)v(x)=1,\quad x\in R.$$

These properties determine $u(x)$ and $v(x)$ up to mutually inverse positive constant factors, and the function $\rho(x)=u(x)v(x)$, $x\in R$ is uniquely determined by $q(x)$, $x\in R$. In the present paper, we obtain an asymptotic formula for computing $\rho(x)$ as $|x|\to\infty$. As an application, under conditions (2), we study the behavior at infinity of solution of the Riccati equation

$$z'(x)+z(x)^2=q(x),\quad x\in R.$$

1. Introduction

In the present paper, we consider an equation

$$y''(x)=q(x)y(x),\quad x\in R \tag{1.1}$$

under assumptions

$$0\le q(x)\in L_1^{\rm loc}(R),\quad \int_{-\infty}^x q(t)dt>0,\quad \int_x^\infty q(t)dt>0\quad \text{for all}\quad x\in R. \tag{1.2}$$

Further, we assume conditions (1.2) are satisfied, without special mention. Our general goal is to study some asymptotic properties (as $|x|\to\infty$) of solution of equations (1.1).

In order to give a concrete statement of the problem, we need the following known facts (see [7, Ch. XI, §6],[2]). First we note that equation (1.1) has a fundamental system of solutions (FSS) $\{u(x),v(x)\}$ which is defined, up to mutually inverse positive constant factors, by the properties

$$v(x)>0,\quad u(x)>0,\quad v'(x)\ge 0,\quad u'(x)\le 0\quad \text{for}\quad x\in R, \tag{1.3}$$

$$v'(x)u(x)-u'(x)v(x)=1\quad \text{for}\quad x\in R, \tag{1.4}$$

$$\lim_{x\to-\infty}\frac{v(x)}{u(x)}=\lim_{x\to\infty}\frac{u(x)}{v(x)}=0, \tag{1.5}$$

$$\int_{-\infty}^0\frac{d\xi}{v(\xi)^2}=\int_0^\infty\frac{d\xi}{u(\xi)^2}=\infty,\quad \int_0^\infty\frac{d\xi}{v(\xi)^2}<\infty,\quad \int_{-\infty}^0\frac{d\xi}{u^2(\xi)}<\infty. \tag{1.6}$$

Relations (1.3)–(1.6) mean that $u(x)$ and $v(x)$ are principal solutions of (1.1) on $(0,\infty)$ and $(-\infty,0)$, respectively (see [7, Ch. XI, §6]. Therefore, we call an FSS $\{(u(x),v(x)\}$ with properties (1.3)–(1.6) a principal FSS (PFSS) of equation (1.1).

The solutions $u(x),v(x)$ from a PFSS of (1.1) are related as follows:

$$u(x)=v(x)\int_x^\infty\frac{dt}{v(t)^2},\quad v(x)=u(x)\int_{-\infty}^x\frac{dt}{u(t)^2},\quad x\in R. \tag{1.7}$$

From (1.7), it follows that the function $\rho(x)$

$$\rho(x)\overset{\text{def}}{=}u(x)v(x)=v(x)^2\int_x^\infty\frac{dt}{v(t)^2}=u(x)^2\int_{-\infty}^x\frac{dt}{u(t)^2},\quad x\in R \tag{1.8}$$

does not depend on the choice of a PFSS and is determined uniquely by equation (1.1), i.e., by the function $q(x)$. Therefore, the Davies-Harrell representation (1.9) for a PFSS of equation (1.1) (see [6]) is very important for the theory of equation (1.1)

$$u(x)=\sqrt{\rho(x)}\exp\left(-\frac12\int_{x_0}^x\frac{d\xi}{\rho(\xi}\right),\quad v(x)=\sqrt{\rho(x)}\exp\left(\frac12\int_{x_0}^x\frac{d\xi}{\rho(x)}\right),\quad x\in R. \tag{1.9}$$

Here x_0 is the unique root of the equation $u(x)=v(x)$ (such an interpretation of Davies-Harrell's formulas was proposed in [2]). Thus for all $x\in R$, any PFSS of (1.1) can be expressed via $\rho(x)$, and the choice of a particular PFSS of (1.1) is determined by the choice of x_0 in (1.9).

Representation (1.9) becomes even more important if one takes into account Otelbaev's a priori inequalities

$$\frac{d(x)}{4}\le\rho(x)\le\frac32 d(x),\quad x\in R. \tag{1.10}$$

Here $d(x)$ is the unique solution in $d\ge0$ of

$$d\int_{x-d}^{x+d}q(t)dt=2,\quad x\in R. \tag{1.11}$$

Remark 1.1. The function $d(x)$ was introduced by M. Otelbaev (see, for example, [9]). It is well-defined (see §2, Lemma 2.1). Inequalities of type (1.10) were first obtained in [8] (under requirements of $q(x)$ stronger than (1.2)), and therefore we relate them and the function $d(x)$ to M. Otelbaev. Note that in [8] another auxiliary function, more complicated than $d(x)$, was used. See [2] for the proof of estimates (1.10) under conditions (1.2).

The study of $\rho(x)$ started in [2] was continued in [4, 3]. In [4] more precise inequalities of type (1.10) were obtained, and in [3], under some additional requirements of (1.2) to $q(x)$, an asymptotic formula for computation of $\rho(x)$ as $|x|\to\infty$, was obtained. We shall need this formula later. To state it, let us introduce the following

Definition 1.2. [3] Suppose that condition (1.2) holds. We say that $q(x)$ belongs to the class H (and write $q(x)\in H$) if there exists a continuous function $k(x)$ in $x\in R$ with properties:

1)
$$k(x)\ge 2,\quad x\in R;\ k(x)\to\infty\quad as\quad |x|\to\infty; \tag{1.12}$$

2) there is an absolute positive constant c_1 such that for all $x\in R$ the following inequalities hold:
$$c_1^{-1}k(x)\le k(t)\le c_1k(x)\quad for\quad t\in[x-k(x)d(x),x+k(x)d(x)] \tag{1.13}$$

3) there is an absolute positive constant c_2 such that for all $x\in R$ the following estimate holds:
$$\Phi(x)\overset{def}{=}k(x)d(x)\sup_{z\in[0,k(x)d(x)]}\left|\int_0^z(q(x+t)-q(x-t))dt\right|\le c_2. \tag{1.14}$$

In the sequel we assume that if $q(x)\in H$, we denote by $k(x)$ the function from Definition 1.2. For example, if $q(x)\in H$, then below by $F(x)$ we denote the function
$$F(x)\overset{\text{def}}{=\!=}\sqrt{k(x)}d(x)\sup_{z\in[0,\sqrt{k(x)}d(x)]}\left|\int_0^z(q(x+t)-q(x-t))dt\right|,\quad x\in R. \tag{1.15}$$
Later we omit the reference to the conditions (1.12), (1.13) and (1.14), which the function $k(x)$ in (1.15) (and in any similar situation) satisfies. By c we denote any absolute positive constants which are not essential for exposition and which may differ even within a single chain of computations. Constants essential for exposition are supplied with indices, as, for example, in Definition 1.2.

Theorem 1.3. [3] *Suppose that $q(x)\in H$ and, in addition, $q(x)\ge 1$ for $x\in R$. Then for all $|x|\gg 1$, we have*
$$|\rho'(x)|\le c[F(x)+\exp(c^{-1}\sqrt{k(x)})]\le\frac{c}{\sqrt{k(x)}}, \tag{1.16}$$

$$\rho(x)=\frac{d(x)}{2}(1+\varepsilon(x)),\quad |\varepsilon(x)|\le c\alpha(x)\le\frac{c}{\sqrt{k(x)}}. \tag{1.17}$$

Here (see (1.15)*):*

$$\alpha(x)=\begin{cases}\exp(-c^{-1}\sqrt{k(x)})+\sup\limits_{t\ge x-d(x)}F(t), & \text{if}\quad x\ge 0\\ \exp(-c^{-1}\sqrt{k(x)})+\sup\limits_{t\le x+d(x)}F(t), & \text{if}\quad x\le 0.\end{cases}\tag{1.18}$$

Here are some comments on Theorem 1.3. The main goal of this statement is to make inequalities (1.10) more precise for $|x|\gg 1$. A solution is suggested in (1.17)–(1.18). Clearly, in view of representations (1.19) of PFSS, formulas of type (1.17)–(1.18) are important for the theory of equation (1.1). In addition, they are applied, for example, in the spectral theory of the Sturm-Liouville operator and in the theory of the Riccati equation (see [4, 3]). Therefore, their further development may be useful for equation (1.1) as well as for its applications. Note that relations (1.17)–(1.18) and (1.10)–(1.11) do not completely agree with one another. In particular, Otelbaev's inequalities are local because to estimate the function $\rho(x)$ in a point $x\in R$, one only uses the values $q(t)$ for all t from the finite segment $[x-d(x),x+d(x)]$. In contrast, asymptotic estimates (1.17) are not local because to estimate $\rho(\cdot)$ in a point x ($|x|\gg 1$) one uses the values $q(t)$ for all t belonging to one of the infinite intervals $(-\infty,x+d(x)]$ or $[x-d(x),\infty)$ (see (1.11) and (1.18)). Analysis of the examples to Theorem 1.3 from [4, 3] shows that in formula (1.17), the estimates of the remainder term $\varepsilon(\cdot)$ in a point x ($|x|\gg 1$) are always formed from the values $q(t)$ related to some local neighborhood of x. This means that in (1.17)–(1.18), when estimating $\varepsilon(x)$, perhaps we impose redundant conditions on the function $q(\cdot)$.

Now, that we have clarified some disadvantages of the relations (1.17)–(1.18), we are able finally to formulate the main goal of this paper: to obtain an asymptotic formula with a local estimate of the remainder term for computing $\rho(x)$ as $|x|\to\infty$. This problem is solved in Theorem 1.4 which is the main result of the present paper, as follows:

Theorem 1.4. *Suppose $q(x)\in H$. Then for all $|x|\gg 1$ estimates* (1.16) *hold, and we have the following relations:*

$$\rho(x)=\frac{d(x)}{2}(1+\varepsilon(x)),\quad |\varepsilon(x)|\le c\beta(x).\tag{1.19}$$

Here

$$\beta(x)=\exp(-c^{-1}\sqrt{k(x)})+\sup_{t\in\Delta(x)}F(x)\le\frac{c}{\sqrt{k(x)}},\quad \Delta(x)=[x-d(x),x+d(x)].\tag{1.20}$$

We give here Theorem 1.5 containing a more detailed variant of formula (1.19) which is intended for the following particular application. We plan to apply (1.19) for constructing

approximations to the solutions of the equation

$$-y''(x) + q(x)y(x) = f(x), \quad x \in R \tag{1.21}$$

with $f(x) \in L_p(R)$, $p \in [1, \infty]$ ($L_\infty(R) := c(R)$) and $q(x) \in H$. To solve this problem, we need a more detailed version of Theorem 1.4. It is convenient to state it here as a separate assertion. First note that the functions $q(x) \in H$ have the following property. For every $q(x) \in H$ there is an absolute positive constant c_3 such that (see §2, Lemma 2.7)

$$c_3^{-1}d(x) \le d(t) \le c_3 d(x) \quad \text{for} \quad |t-x| \le \sqrt{k(x)}d(x), \quad x \in R. \tag{1.22}$$

We introduce some notation:

$$d_0 = \sup_{x\in R} d(x), \tag{1.23}$$

$$\eta_1(x) = 4[F(x) + \sqrt{c_3}\exp(-(3c_3)^{-1}\sqrt{k(x)})], \quad x \in R, \tag{1.24}$$

$$\eta_2(x) = 65\left[\sup_{t\in\Delta(x)} F(t) + \sqrt{c_3}\exp\left(-(3c_3\sqrt{c_3})^{-1}\sqrt{k(x)}\right)\right], \quad x \in R. \tag{1.25}$$

Theorem 1.5. *Suppose $q(x) \in H$ and $d_0 < \infty$. Denote by S_0 a point on the number axis such that for all $|x| \ge s_0$ the following inequalities hold (see* (1.12) *and* (1.16)*):*

$$k(x) \ge 64c_2^2, \quad \eta_1(x) \le 10^{-3}, \tag{1.26}$$

and set $s_1 = s_0 + d_0 + 1$. Then the following relations hold:

$$|\rho'(x)| \le \eta_1(x) \quad \textit{for} \quad |x| \ge s_1, \tag{1.27}$$

$$\rho(x) = \frac{d(x)}{2}(1+\varepsilon(x)), \quad |\varepsilon(x)| \le \eta_2(x) \quad \textit{for} \quad |x| \ge s_1. \tag{1.28}$$

Remark 1.6. Theorems 1.4 and 1.5 will be proved together in §§2–4. Each section contains a separate part of the proof accompanied by necessary comments. The proof of formula (1.19) for a concrete equation, along with technical details, is contained in §8.

Let us now compare Theorems 1.3 and 1.4. Theorem 1.3 contains the requirement $q(x) \ge 1$ for $x \in R$ which is not contained in Theorem 1.4. This restriction is not essential since relations (1.17)–(1.18) can also be obtained by the method of [3] using only condition (1.2) and the condition $q(x) \in H$. Nevertheless, in order to reveal the principal difference between Theorems 1.3 and 1.4, in our comments below we assume that the condition $q(x) \ge 1$, $x \in R$ holds true. This convention immediately implies that a new "local" version of (1.19)–(1.20) of

asymptotic estimates at infinity for the function $\rho(x)$ is obtained under the same assumptions on which, in Theorem 1.3 only guaranteed a "non-local" form of estimates (1.17)–(1.18). This "qualitative" advantage of Theorem 1.4 with respect to Theorem 1.3 is evidently important for solving theoretical problems related to the properties of the function $\rho(x)$ at infinity. However, when applied to concrete equations, a "general quantitative advantage" of relations (1.19)–(1.20) with respect to (1.17)–(1.18) turns out to be more important.

This advantage can be expressed as follows: the asymptotic formula (1.19)–(1.20) can be viewed as a refinement of the asymptotic formula (1.17)–(1.18) in the class H. To justify that, note that Theorems 1.3 and 1.4 differ only in the functions $\alpha(x)$ and $\beta(x)$ which give an estimate of the same remainder terms $\varepsilon(x)$ in the asymptotic formula

$$\rho(x) = \frac{d(x)}{2}(1+\varepsilon(x)), \qquad \lim_{|x|\to\infty} \varepsilon(x) = 0. \tag{1.29}$$

Here both functions are constructed by the function $q(x)$, $x \in R$, are continuous for $x \in R$ and satisfy the relations

$$0 < \beta(x) \le \alpha(x), \quad x \in R, \qquad \lim_{|x|\to\infty} \alpha(x) = \lim_{|x|\to\infty} \beta(x) = 0. \tag{1.30}$$

Denote

$$L = \sup_{q(x)\in H} \overline{\lim_{|x|\to\infty}} \frac{\alpha(x)}{\beta(x)}. \tag{1.31}$$

We say that the asymptotic formulas (1.17)–(1.18) and (1.19)–(1.20) are equivalent in the class H if $L < \infty$. If $L = \infty$, we say that the asymptotic formula (1.19)–(1.20) is a refinement of the asymptotic formula (1.17)–(1.18) in the class H. With this terminology, the following assertion give the main relationship between Theorems 1.3 and 1.4.

Theorem 1.7. *The asymptotic formula* (1.19)–(1.20) *is a refinement of the asymptotic formula* (1.17)–(1.18) *in the class H.*

We give here an example of an application of Theorem 1.4. Consider a Riccati equation

$$y'(x) + y(x)^2 = q(x), \quad x \in R. \tag{1.32}$$

In §6, we prove the following theorem which complements one of the results of [4].

Theorem 1.8. *Suppose $q(x) \in H$. Then the following assertions hold:*

A) *There exists a unique solution* $y_1(x)$ $(y_2(x))$ *of equation* (1.32) *defined for all* $x \in R$ *and satisfying the equalities*

$$\lim_{x\to-\infty} y_1(x)d(x) = \lim_{x\to\infty} y_1(x)d(x) = -1$$
$$\left(\lim_{x\to-\infty} y_2(x)d(x) = \lim_{x\to\infty} y_2(x)d(x) = 1\right). \tag{1.33}$$

B) *Let* $y_+(x)$ *be a solution of* (1.32) *defined on* $[c,\infty)$ *for some* c. *Then* $y_+(x) \neq y_1(x)$ *if and only if*

$$\lim_{x\to\infty} y_+(x)d(x) = 1. \tag{1.34}$$

C) *Let* $y_-(x)$ *be a solution of* (1.32) *defined on* $(-\infty, c]$ *for some* c. *Then* $y_-(x) \neq y_2(x)$ *if and only if*

$$\lim_{x\to-\infty} y_-(x)d(x) = -1. \tag{1.35}$$

Note that an example of Theorem 1.8 is contained in §8.

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2. Technical assertions

In this section, we present some auxiliary assertions on the properties of the function $d(x)$ (see (1.11)). Most of these lemmas were obtained in [3] under the assumption

$$1 \le q(x) \in L_1^{\text{loc}}(R), \quad x \in R. \tag{2.1}$$

To pass from condition (2.1) to condition (1.2), we have to prove that the "old" assertions remain true under the "new" assumptions. Our new proofs are simpler and shorter than the previous ones and significantly differ from those presented in [3].

Lemma 2.1. [9, Ch.I, §5] *For every given* $x \in R$, *equation* (1.1) *has a unique solution in* $d \ge 0$.

Proof. The functions

$$\varphi_1(d) = \frac{2}{d}, \quad \varphi_2(d) = \int_{x-d}^{x+d} q(t)dt, \quad d \in (0,\infty)$$

have the following properties:

1) the function $\varphi_1(d)$ is monotone decreasing from infinity to zero on $(0,\infty)$;
2) the function $\varphi_2(d)$ is non-decreasing and non-negative on $(0,\infty)$ and, in addition, $\lim_{d\to\infty} \varphi_2(d) = \infty$ (see (1.2)).

From 1)–2) and the continuity of the two functions, it follows that their graphs intersect at one point. ☐

Lemma 2.2. *For every $x \in R$, the inequality $\eta \geq d(x)$ $(0 \leq \eta \leq d(x))$ holds if and only if*

$$S(\eta) \geq 1 \ (S(\eta) \leq 2), \quad S(\eta) \overset{def}{=} \eta \int_{x-\eta}^{x+\eta} q(t)dt. \tag{2.2}$$

Proof of Lemma 2.2. Necessity.

If $\eta \geq d(x)$, then $S(\eta) \geq S(d(x)) = 2$.

Proof of Lemma 2.2. Sufficiency.

Assume the contrary: $S(\eta) \geq 2$, but $\eta < d(x)$. Then $2 \leq S(\eta) \leq S(d(x)) = 2 \Rightarrow S(\eta) = 2$. Hence $\eta = d(x)$ by Lemma 2.1. Contradiction. ☐

For a given $x \in R$, consider an equation in $d \geq 0$:

$$G(d) = 1, \qquad G(d) \overset{\text{def}}{=} \int_0^d \int_{x-t}^{x+t} q(\xi)d\xi dt, \qquad d \geq 0. \tag{2.3}$$

Lemma 2.3. *For every $x \in R$, equation (2.3) has a unique positive solution. Denote it by $\hat{d}(x)$. The function $\hat{d}(x)$ satisfies the inequalities*

$$d(x) \leq 2\hat{d}(x) \leq 3d(x), \quad x \in R, \tag{2.4}$$

$$1 \leq \hat{d}(x) \int_{x-\hat{d}(x)}^{x+\hat{d}(x)} q(t)dt, \quad x \in R. \tag{2.5}$$

In addition, $\hat{d}(x)$ has a continuous derivative for $x \in R$, and

$$|\hat{d}'(x)| \leq \hat{d}(x) \left| \int_0^{\hat{d}(x)} (q(x+t) - q(x-t))dt \right|, \quad x \in R. \tag{2.6}$$

Remark 2.4. The function $\hat{d}(x)$ was introduced in [1] under condition (2.1).

Proof. Clearly, $G(d)$ is continuous for all $d \geq 0$. In addition, $G(0) = 0$ and $G(d) \to \infty$ as $d \to \infty$ since (see (1.2))

$$G(d) \geq \int_{d/2}^d \int_{x-t}^{x+t} q(\xi)d\xi dt \geq \frac{d}{2} \int_{x-d/2}^{x+d/2} q(\xi)d\xi \to \infty \quad \text{as} \quad d \to \infty.$$

Since we have, in addition,

$$G'(d) = \int_{x-d}^{x+d} q(\xi)d\xi \geq 0, \quad x \in R, \quad d \geq 0, \tag{2.7}$$

equation (2.3) has at least one solution $d_0 > 0$.

The obvious relations (see (2.7))

$$1 = G(d_0) = \int_0^{d_0}\int_{x-t}^{x+t} q(\xi)d\xi dt \le d_0 \int_{x-d_0}^{x+d_0} q(\xi)d\xi = d_0 G'(d_0) \tag{2.8}$$

implies that $G'(d_0) > 0$, and therefore d_0 is a unique root of (2.3). Denote it by $\hat{d}(x)$. From (2.8) and Lemma 2.2, it follows that $d(x) \le 2\hat{d}(x)$ since

$$2 = 2G(d(x)) \le 2S(\hat{d}(x)) \le S(2\hat{d}(x)).$$

The second inequality in (2.4) also follows from Lemma 2.2:

$$2 = 2G(\hat{d}(x)) \ge 2\int_{2\hat{d}(x)/3}^{\hat{d}(x)}\int_{x-t}^{x+t} q(\xi)d\xi dt = \frac{2}{3}\hat{d}(x)\int_{x-2\hat{d}(x)/3}^{x+2\hat{d}(x)/3} q(\xi)d\xi = S\left(\frac{2\hat{d}(x)}{3}\right).$$

Finally, the estimate (2.5) coincides with (2.8), and it remains to check (2.6). Let us regard $\hat{d}(x)$ as an implicit function, i.e., as the positive solution of the equation

$$F(x,z) = \int_0^z \int_{x-t}^{x+t} q(\xi)d\xi dt - 1 = 0. \tag{2.9}$$

In a neighborhood of the point $(x, \hat{d}(x))$, the function $F(x,z)$ is continuous together with its partial derivatives

$$F'_x(x,z) = \int_0^z (q(x+t) - q(x-t))dt, \quad F'_z(x,z) = \int_{x-z}^{x+z} q(\xi)d\xi.$$

In addition, according to (2.6), we have

$$F'_z(x,z)\,\big|_{z=\hat{d}(x)} = \int_{x-\hat{d}(x)}^{x+\hat{d}(x)} q(\xi)d\xi \ge \frac{1}{\hat{d}(x)} > 0.$$

Hence $\hat{d}(x)$ is differentiable, and

$$0 = \hat{d}'(x)\int_{x-\hat{d}(x)}^{x+\hat{d}(x)} q(\xi)d\xi + \int_0^{\hat{d}(x)} (q(x+t) - q(x-t))dt, \quad x \in R. \tag{2.10}$$

From (2.10) and (2.5), it now follows that

$$\frac{|\hat{d}'(x)|}{\hat{d}(x)} \le |\hat{d}'(x)|\int_{x-\hat{d}(x)}^{x+\hat{d}(x)} q(\xi)d\xi = \left|\int_0^{\hat{d}(x)} (q(x+t) - q(x-t))dt\right|.$$

□

Corollary 2.5. *If* $q(x) \in H$, *then*

$$2k(x)|\hat{d}'(x)| \le 3c_2. \tag{2.11}$$

Proof. From (1.12) and (2.4), we get

$$\hat{d}(x) \le \frac{3}{2}d(x) \le k(x)d(x), \quad x \in R. \tag{2.12}$$

Therefore, according to (2.16), (1.14) and (2.12), we have

$$\begin{aligned}|\hat{d}'(x)| &\le \frac{3}{2}d(x)\left|\int_0^{\hat{d}(x)}(q(x+t)-q(x-t))dt\right|\\ &\le \frac{3}{2}\frac{1}{k(x)}\left[k(x)d(x)\sup_{z\in[0,k(x)d(x)]}\left|\int_0^z(q(x+t)-q(x-t))dt\right|\right]\le\frac{3c_2}{2k(x)}.\end{aligned}$$

□

In what follows, we often us an obvious general assertion which, for convenience, will be stated as a separate lemma.

Lemma 2.6. *Let $\varphi(x)$ and $\psi(x)$ be positive and continuous for functions $f\in R$. If there exists an interval (a,b) such that*

$$c^{-1}\varphi(x)\le\psi(x)\le c\varphi(x)\quad\textit{for all}\quad x\notin(a,b), \tag{2.13}$$

then inequalities (2.13) *remain true for all $x\in R$ (perhaps after replacing with a larger constant).*

Proof. The function $f(x)=\frac{\psi(x)}{\varphi(x)}$ is continuous and positive for $x\in[a,b]$. Hence its minimum m and maximum M on $[a,b]$ are finite positive numbers. Let $\tilde{c}=\max\{c,m^{-1},M\}$ where c is the constant from (2.13). Then $\tilde{c}^{-1}\varphi(x)\le\psi(x)\le\tilde{c}\varphi(x)$ for $x\in R$. □

Lemma 2.7. *Let $q(x)\in H$ and*

$$\omega(x)=[\omega^-(x),\omega^+(x)]=\left[x-\sqrt{k(x)}d(x),x+\sqrt{k(x)}d(x)\right],\quad x\in R. \tag{2.14}$$

Then there exists an absolute positive constant c_3 such that for all $x\in R$ and $t\in\omega(x)$, the following inequalities hold:

$$c_3^{-1}d(x)\le d(t)\le c_3d(x). \tag{2.15}$$

Proof. By (1.12), there is $x_0\gg1$ such that $k(x)\ge36(c_1c_2)^2$ for $|x|\ge x_0$ (see (1.13) and (1.14)).

In the following relations, we assume that $|x|\ge x_0$, $t\in\omega(x)$ and use (2.11), (1.13) and (2.4):

$$\begin{aligned}|\hat{d}(t)-\hat{d}(x)&=\left|\int_x^t\hat{d}'(\xi)d\xi\right|\le\left|\int_x^t|\hat{d}'(\xi)|d\xi\right|\le\frac{3c_2}{2}\left|\int_x^t\frac{d\xi}{k(\xi)}\right|\\ &\le\frac{3c_1c_2}{2}\frac{|t-x|}{k(x)}\le\frac{3c_1c_2}{2}\frac{d(x)}{\sqrt{k(x)}}\le\frac{3c_1c_2}{\sqrt{k(x)}}\hat{d}(x)\le\frac{\hat{d}(x)}{2}\\ &\Rightarrow\quad 2^{-1}\hat{d}(x)\le\hat{d}(t)\le2\hat{d}(x)\quad\text{for}\quad t\in\omega(x),\quad|x|\ge x_0.\end{aligned} \tag{2.16}$$

Denote

$$\varphi(x)=\hat d(x),\qquad \psi_1(x)=\min_{t\in\omega(x)}\hat d(t),\qquad \psi_2(x)=\max_{t\in\omega(x)}\hat d(t),\qquad x\in R. \tag{2.17}$$

With the notation of (2.17), inequalities (2.16) have the following form:

$$2^{-1}\varphi(x)\le\psi_1(x),\psi_2(x)\le 2\varphi(x)\qquad\text{for}\quad |x|\ge x_0. \tag{2.18}$$

According to (2.18), from Lemma 2.6 it follows that there exists a constant $\tilde c$ such that

$$\tilde c^{-1}\varphi(x)\le\psi_1(x),\psi_2(x)\le \tilde c\varphi(x)\qquad\text{for}\quad x\in R. \tag{2.19}$$

The estimates (2.19) immediately imply the inequalities

$$\tilde c^{-1}\hat d(x)\le \hat d(t)\le \tilde c\hat d(x)\qquad\text{for}\quad t\in\omega(x),\quad x\in R. \tag{2.20}$$

The relations (2.15) with $c_3=3\tilde c$ follow from (2.20) and (2.4):

$$\frac{d(t)}{d(x)}=\frac{d(t)}{\hat d(t)}\cdot\frac{\hat d(t)}{\hat d(x)}\cdot\frac{\hat d(x)}{d(x)}\le 2\cdot\tilde c\cdot\frac32=3\tilde c\qquad\text{for}\quad t\in\omega(x),\quad x\in R,$$

$$\frac{d(t)}{d(x)}=\frac{d(t)}{\hat d(t)}\cdot\frac{\hat d(t)}{\hat d(x)}\cdot\frac{\hat d(x)}{d(x)}\ge \frac23\cdot\frac1{\tilde c}\cdot\frac12=\frac1{3\tilde c}\qquad\text{for}\quad t\in\omega(x),\quad x\in R.$$

□

Lemma 2.8. *Under condition* (1.2), *we have*

$$\lim_{x\to-\infty}(x+d(x))=-\infty,\qquad \lim_{x\to\infty}(x-d(x))=\infty. \tag{2.21}$$

Proof. The equalities in (2.21) are checked in a similar way. Let us prove, for example, the second one. We show that

$$\underline{\lim}_{x\to\infty}(x-d(x))=\infty. \tag{2.22}$$

Assume the contrary. Then there exists a number $a\in R$ and a sequence $\{x_n\}_{n=1}^\infty$ such that

$$x_n-d(x_n)\le a\quad\text{for}\quad n\in N\qquad\text{and}\qquad x_n\to\infty\quad\text{as}\quad n\to\infty. \tag{2.23}$$

From (2.23) it follows that there is $n_0\gg 1$ such that for all $n\ge n_0$, the following inequalities hold:

$$d(x_n)\ge x_n-a=x_n\left(1-\frac{a}{x_n}\right)\ge\frac{x_n}{2},\qquad n\ge n_0. \tag{2.24}$$

Then using (2.23), (2.24) and (1.11), we get

$$2=d(x_n)\int_{x_n-d(x_n)}^{x_n+d(x_n)}q(t)dt\ge\frac{x_n}{2}\int_a^{x_n}q(t)dt\quad\Rightarrow\quad\frac{4}{x_n}\ge\int_a^{x_n}q(\xi)d\xi. \tag{2.25}$$

Clearly, (2.23) and (2.25) contradict (1.2), which leads to (2.22). But this implies the statement of the lemma because

$$\infty = \varliminf_{x\to\infty}(x-d(x)) \le \varlimsup_{x\to\infty}(x-d(x)) \le \infty \quad \Rightarrow \quad \varliminf_{x\to\infty}(x-d(x)) = \varlimsup_{x\to\infty}(x-d(x)) = \infty.$$

□

3. Main asymptotic formula

In this section, our goal is to prove the following assertion.

Lemma 3.1. *Suppose that condition* (1.2) *holds and*

$$\lim_{|x|\to\infty}\rho'(x) = 0. \tag{3.1}$$

Then we have

$$\rho(x) = \frac{d(x)}{2}(1+\varepsilon(x)), \qquad \lim_{|x|\to\infty}\varepsilon(x) = 0, \tag{3.2}$$

and the following relations hold:

$$|\varepsilon(x)| \le ch(x), \qquad h(x)\to 0 \qquad as \quad |x|\to\infty. \tag{3.3}$$

Here

$$h(x) = \sup_{t\in\Delta(x)}|\rho'(t)|, \qquad \Delta(x) = [\Delta^-(x),\Delta^+(x)] = [x-d(x), x+d(x)], \qquad x\in R. \tag{3.4}$$

Denote by z_0 a point on the number axis such that (see (3.1))

$$|\rho'(x)| \le 10^{-3} \qquad \text{for all} \qquad |x|\ge z_0. \tag{3.5}$$

Suppose that in addition to (1.2), (3.1), we have $d_0<\infty$ (see (1.23)). Then equality (3.2) holds, and

$$|\varepsilon(x)| \le 18h(x) \qquad \text{for} \qquad |x|\ge z_1, \qquad z_1 \overset{\text{def}}{=} z_0+d_0+1. \tag{3.6}$$

To prove Lemma 3.1, we need the following auxiliary assertions.

Lemma 3.2. *For* $x\in R$ *the following relations hold:*

$$|\rho'(x)| < 1, \tag{3.7}$$

$$\frac{v'(x)}{v(x)} = \frac{1+\rho'(x)}{2\rho(x)}, \qquad \frac{u'(x)}{u(x)} = -\frac{1-\rho'(x)}{2\rho(x)}. \tag{3.8}$$

Proof. Let us show that (see (1.3))

$$v'(x) > 0, \qquad u'(x) < 0 \qquad \text{for} \qquad x \in R. \tag{3.9}$$

For a given $x \in R$, by (1.2) there exists $a \in (-\infty, x]$ such that

$$\int_a^x q(t)dt > 0.$$

Then from (1.2) and (1.3) it follows that

$$v'(x) = v'(a) + \int_z^x q(t)v(t)dt \geq \int_a^x q(t)v(t)dt \geq v(a)\int_a^x q(t)dt > 0.$$

The second inequality from (3.9) can be checked in a similar way. To prove (3.8), it suffices to differentiate (1.9). Inequality (3.7) follows from (3.8) and (3.9). □

Lemma 3.3. *For $x \in R$, we have*

$$\frac{1+\rho'(\Delta^+(x))}{1-\rho'(\Delta^+(x))} \cdot \frac{1-\rho'(\Delta^-(x))}{1+\rho'(\Delta^-(x))} = \exp\left(4\int_{\Delta(x)} \frac{q(t)\rho(t)dt}{1-\rho'(t)^2} - \int_{\Delta(x)} \frac{dt}{\rho(t)}\right). \tag{3.10}$$

Here $\Delta(x) = [\Delta^-(x), \Delta^+(x)] = [x - d(x), x + d(x)]$.

Proof. From (3.9) and (1.1) for $t \in R$, it follows that

$$v''(t) = q(t)v(t) \quad \Rightarrow \quad \frac{v''(t)}{v'(t)} = q(t)\frac{v(t)}{v'(t)} \quad \Rightarrow \quad ln\frac{v'(\Delta^+(x))}{v'(\Delta^-(x))} = \int_{\Delta(x)} \frac{q(t)v(t)dt}{v'(t)},$$

$$u''(t) = q(t)u(t) \quad \Rightarrow \quad \frac{u''(t)}{u'(t)} = q(t)\frac{u(t)}{u'(t)} \quad \Rightarrow \quad ln\frac{u'(\Delta^+(x))}{u'(\Delta^-(x))} = \int_{\Delta(x)} \frac{q(t)u(t)dt}{u'(t)}.$$

These inequalities imply

$$\frac{v'(\Delta^+(x))}{v'(\Delta^-(x))} \cdot \frac{u'(\Delta^-(x))}{u'(\Delta^+(x))} = \exp\left(\int_{\Delta(x)} q(t)\left(\frac{v(t)}{v'(t)} - \frac{u(t)}{u'(t)}\right)dt\right), \quad x \in R. \tag{3.11}$$

When substituting (3.8) into (3.11), we get

$$\frac{1+\rho'(\Delta^+(x))}{1-\rho'(\Delta^+(x))} \cdot \frac{1-\rho'(\Delta^-(x))}{1+\rho'(\Delta^-(x))} \cdot \frac{v(\Delta^+(x))}{u(\Delta^+(x))} \cdot \frac{u(\Delta^-(x))}{v(\Delta^-(x))} = \exp\left(4\int_{\Delta(x)} \frac{q(t)\rho(t)dt}{1-\rho'(t)^2}\right). \tag{3.12}$$

Furthermore, according to (1.9) we have

$$\frac{v(\Delta^+(x))}{u(\Delta^+(x))} = \exp\left(\int_{x_0}^{\Delta^+(x)} \frac{dt}{\rho(t)}\right), \qquad \frac{v(\Delta^-(x))}{u(\Delta^-(x))} = \exp\left(\int_{x_0}^{\Delta^-(x)} \frac{dt}{\rho(t)}\right), \qquad x \in R. \tag{3.13}$$

To prove (3.10), it remains to substitute (3.13) into (3.12). □

Lemma 3.4. *Suppose that condition* (1.2) *holds. Then*

$$\rho(t) \leq \frac{5}{2}d(x) \qquad \text{for} \qquad t \in \Delta(x) = [x - d(x), x + d(x)], \qquad x \in R. \tag{3.14}$$

Proof. By Lagrange's formula,

$$\rho(t)=\rho(x)+\rho'(\xi)(t-x),\qquad t\in\Delta(x),\qquad x\in R. \tag{3.15}$$

The point ξ in (3.15) lies between t and x. Then (3.15), together with (3.7) and (3.10), lead to (3.14):

$$\rho(t)\le\rho(x)+|\rho'(\xi)|\,|t-x|\le\rho(x)+d(x)\le\frac{3}{2}d(x)+d(x)=\frac{5}{2}d(x).$$

□

In the sequel, we assume that conditions (1.2) and (3.1) hold and do not mention them in the statements.

Lemma 3.5. *For all $|x|\gg1$, the following inequalities hold:*

$$\left|4\int_{\Delta(x)}\frac{q(t)\rho(t)dt}{1-\rho'(t)^2}-8\frac{\rho(x)}{d(x)}\right|\le 8,0201h(x). \tag{3.16}$$

In addition, if $d_0<\infty$ (see (1.23)), then (3.16) holds for all $|x|\ge z_1$ (see (3.6)).

Proof. In the following transformations, we use the definition of $d(x)$ (see (1.11)):

$$\begin{aligned}4\int_{\Delta(x)}\frac{q(t)\rho(t)dt}{1-\rho'(t)^2}&=4\int_{\Delta(x)}q(t)\rho(t)dt+4\int_{\Delta(x)}\frac{q(t)\rho(t)\rho'(t)^2}{1-\rho'(t)^2}dt\\&=4\rho(x)\int_{\Delta(x)}q(t)dt+4\int_{\Delta(x)}q(t)(\rho(t)-\rho(x))dt+4\int_{\Delta(x)}\frac{q(t)\rho(t)\rho'(t)^2}{1-\rho'(t)^2}dt\\&=\frac{8\rho(x)}{d(x)}+4\int_{\Delta(x)}q(t)(\rho(t)-\rho(x))dt+4\int_{\Delta(x)}\frac{q(t)\rho(t)\rho'(t)^2}{1-\rho'(t)^2}dt,\ x\in R.\end{aligned} \tag{3.17}$$

Below, in the estimate of the first integral of (3.17), we use (3.15) and the definitions of $h(x)$, $\Delta(x)$ and $d(x)$ (see (3.4), (1.11)):

$$\begin{aligned}4\left|\int_{\Delta(x)}q(t)(\rho(t)-\rho(x))dt\right|&\le4\int_{\Delta(x)}q(t)|\rho(t)-\rho(x)|dt=4\int_{\Delta(x)}q(t)|\rho'(\xi)|\,|t-x|dt\\&\le4h(x)d(x)\int_{\Delta(x)}q(t)dt=8h(x),\qquad x\in R.\end{aligned} \tag{3.18}$$

Let us estimate the second integral from (3.17). From (2.21), it follows that there is $\tilde z_0\gg z_0$ (see (3.5)) such that

$$\Delta(x)\cap[-z_0,z_0]=\emptyset\qquad\text{for}\quad |x|\gg\tilde z_0. \tag{3.19}$$

In particular, if $d_0<\infty$ (see (1.23)), then one can set $\tilde z_0:=z_1=z_0+d_0+1$ (see (3.6)). Indeed, with such a choice of $\tilde z_0$, we have

1) if $x\le-\tilde z_0\ \Rightarrow\ x+d(x)\le-\tilde z_0+d(x)=-z_0-1+d(x)-d_0<-z_0\ \Rightarrow$ (3.19)
2) if $x\ge-\tilde z_0\ \Rightarrow\ x-d(x)\ge\tilde z_0-d(x)=z_0+1+d_0-d(x)>z_0\ \Rightarrow$ (3.19).

Below, for $|x|\ge \tilde z_0$, we use (3.19), (3.14), (3.5) and (1.11):

$$0\le 4\int_{\Delta(x)}\frac{q(t)\rho(t)\rho'(t)^2dt}{1-\rho'(t)^2}\le \frac{4\cdot 10^{-3}}{1-10^{-6}}h(x)\int_{\Delta(x)}q(t)\rho(t)dt$$
$$\le \frac{4h(x)}{10^3-10^{-3}}\frac{5}{2}d(x)\int_{\Delta(x)}q(t)dt=\frac{20h(x)}{10^3-10^{-3}}\le 0.0201h(x). \tag{3.20}$$

From (3.20) and (3.18), we get (3.16) □

Lemma 3.6. *For all $|x|\gg 1$, we have*

$$\left|\int_{\Delta(x)}\frac{dt}{\rho(t)}-2\frac{d(x)}{\rho(x)}\right|\le 32.16h(x). \tag{3.21}$$

In addition, if $d_0<\infty$ (see (1.23)), then (3.21) holds for all $|x|\ge z_1$ (see (3.6)).

Proof. Let $\tilde z_0$ be the number from Lemma 3.5. In the following transformation, we use (3.15):

$$\int_{\Delta(x)}\frac{dt}{\rho(t)}=\frac{1}{\rho(x)}\int_{\Delta(x)}\frac{\rho(x)dt}{\rho(x)+\rho'(\xi)(t-x)}=\frac{1}{\rho(x)}\int_{\Delta(x)}\frac{dt}{1+\rho'(\xi)\frac{t-x}{\rho(x)}}. \tag{3.22}$$

Consider the integrand in (3.22). Let us check the estimate

$$|\gamma(x,\xi,t)|\le 4\cdot 10^{-3},\quad \gamma(x,\xi,t)\overset{\text{def}}{=}\rho'(\xi)\frac{t-x}{\rho(x)},\quad \xi,t\in\Delta(x),\quad |x|\ge\tilde z_0. \tag{3.23}$$

Indeed, for $|x|\ge\tilde z_0$ from (1.10), it follows that

$$|\gamma(x,\xi,t)|=|\rho'(\xi)|\frac{|t-x|}{\rho(x)}\le 10^{-3}\frac{dx}{\rho(x)}\le 4\cdot 10^{-3}\ \Rightarrow\ (3.23).$$

Below, for $|x|\ge\tilde z_0$, we use (3.23), (1.10) and the definition of $h(x)$ (see (3.4)):

$$\frac{|\gamma(x,\xi,t)|}{|1+\gamma(x,\xi,t)|}\le\frac{|\gamma(x,\xi,t)|}{1-|\gamma(x,\xi,t)|}\le\frac{h(x)}{1-4.10^{-3}}\frac{d(x)}{\rho(x)}$$
$$\le\frac{4000}{996}h(x)\le 4.02h(x),\quad |x|\ge\tilde z_0. \tag{3.24}$$

To finish the proof of (3.21), it remains to apply (3.22), (3.24) and (1.10) for $|x|\ge\tilde z_0$:

$$\left|\int_{\Delta(x)}\frac{dt}{\rho(t)}-2\frac{d(x)}{\rho(x)}\right|=\left|\frac{1}{\rho(x)}\int_{\Delta(x)}\frac{dt}{1+\gamma(x,\xi,t)}-2\frac{d(x)}{\rho(x)}\right|$$
$$=\frac{1}{\rho(x)}\left|\int_{\Delta(x)}\left(\frac{1}{1+\gamma(x,\xi,t)}-1\right)dt\right|$$
$$\le\frac{1}{\rho(x)}\int_{\Delta(x)}\frac{|\gamma(x,\xi,t)|dt}{|1+\gamma(x,\xi,t)|}\le 8.04h(x)\frac{d(x)}{\rho(x)}\le 32.16h(x).$$

□

Proof of Lemma 3.1. Throughout the sequel, we assume $|x| \geq \tilde{z}_0$ where $\tilde{z}_0$ is the number from Lemma 3.5. Consider (3.10). In the following estimates, we use (3.5) and the definition of $h(x)$ (see (3.4)):

$$\left|\frac{1+\rho'(\Delta^+(x))}{1-\rho'(\Delta^+(x))}\cdot\frac{1-\rho'(\Delta^-(x))}{1+\rho'(\Delta^-(x))}-1\right| = \frac{2|\rho'(\Delta^+(x))-\rho'(\Delta^-(x))|}{|+\Delta'(\Delta^-(x))-\rho'(\Delta^+(x))-\rho'(\Delta^-(x))\rho'(\Delta^+(x))|}$$
$$\leq \frac{4h(x)}{1-2.10^{-3}-10^{-6}} \leq 4.009h(x)$$
$$\Rightarrow \frac{1+\rho'(\Delta^+(x))}{1-\rho'(\Delta^-(x))}\,\frac{1-\rho'(\Delta^-(x))}{1+\rho'(\Delta^-(x))} = 1+\delta_1(x), \quad |\delta_1(x)| \leq 4.009h(x), \quad |x| \geq \tilde{z}_0. \tag{3.25}$$

Below, in the transformation of the exponent in (3.10), we use inequalities (3.16) and (3.21):

$$\left|\left(4\int_{\Delta(x)}\frac{q(t)\rho(t)dt}{1-\rho'(t)^2}-\int_{\Delta(x)}\frac{dt}{\rho(t)}\right)-\left(\frac{8\rho(x)}{d(x)}-\frac{2d(x)}{\rho(x)}\right)\right|$$
$$\leq \left|4\int_{\Delta(x)}\frac{q(t)\rho(t)dt}{1-\rho'(t)^2}-\frac{8\rho(x)}{d(x)}\right|+\left|\int_{\Delta(x)}\frac{dt}{\rho(t)}-\frac{2d(x)}{\rho(x)}\right| \leq (8,0201+32.16)h(x) \leq 40.2h(x)$$
$$\Rightarrow 4\int_{\Delta(x)}\frac{q(t)\rho(t)dt}{1-\rho'(t)^2}-\int_{\Delta(x)}\frac{dt}{\rho(t)} = \frac{8\rho(x)}{d(x)}-\frac{2d(x)}{\rho(x)}+\delta_2(x),$$
$$|\delta_2(x)| \leq 40.2h(x), \quad |x| \geq \tilde{z}_0. \tag{3.26}$$

Thus, (see (3.25) and (3.26)) equality (3.10) is reduced to

$$1+\delta_1(x) = \exp\left(8\frac{\rho(x)}{d(x)}-2\frac{d(x)}{\rho(x)}+\delta_2(x)\right), \qquad |x| \geq \tilde{z}_0. \tag{3.27}$$

From Lagrange's formula, (3.5) and (3.25) it follows that

$$ln(1+\delta_1(x)) = \frac{\delta_1(x)}{1+\xi}, \qquad \xi \in (-|\delta_1(x)|, |\delta(x)|)$$
$$\Rightarrow ln(1+\delta_1(x)) = \delta_3(x), \quad |\delta_3(x)| \leq \frac{|\delta_1(x)|}{1-|\delta_1(x)|} \leq \frac{4.009h(x)}{1-4.009\cdot 10^{-3}} < 4.03h(x). \tag{3.28}$$

According to (3.27) and (3.28), we now obtain

$$\delta_3(x) = \frac{8\rho(x)}{d(x)}-\frac{2d(x)}{\rho(x)}+\delta_2(x), \quad |x| \geq \tilde{z}_0$$
$$\Rightarrow \frac{8\rho(x)}{d(x)}-\frac{2d(x)}{\rho(x)} = \delta_4(x), \quad |\delta_4(x)| \leq |\delta_2(x)|+|\delta_3(x)| \leq 44.23h(x), \quad |x| \geq \tilde{z}_0. \tag{3.29}$$

Let us rewrite (3.29) in the following way:

$$\rho(x)^2 = \frac{d(x)^2}{4}\left(1+\delta_4(x)\frac{\rho(x)}{2d(x)}\right), \quad |\delta_4(x)| \leq 44.23h(x), \quad |h| \geq \tilde{z}_0. \tag{3.30}$$

Denote

$$\alpha(x) = \delta_4(x)\frac{\rho(x)}{2d(x)}, \qquad |x| \geq \tilde{z}_0. \tag{3.31}$$

Below, in the estimate of $|\alpha(x)|$, we use (3.30) and (1.10):

$$|\alpha(x)| \le |\delta_4(x)|\frac{\rho(x)}{2d(x)} \le 44.23 \cdot \frac{3}{4}h(x) \le 33.2h(x) \le 0.0332. \tag{3.32}$$

Therefore, from (3.30), (3.31) and (3.32), we get

$$\rho(x) = \frac{d(x)}{2}\sqrt{1+\alpha(x)}, \qquad |x| \ge \tilde{z}_0. \tag{3.33}$$

Furthermore, since

$$\sqrt{1+\nu} = 1 + \frac{\nu}{2} - \frac{1}{2}\left(\frac{\nu}{1+\sqrt{1+\nu}}\right)^2 \qquad \text{for} \quad 1+\nu \ge 0, \tag{3.34}$$

from (3.32) and (3.34), we get

$$\sqrt{1+\alpha(x)} = 1 + \varepsilon(x), \quad |\varepsilon(x)| \le \frac{|\alpha(x)|}{2} + \frac{1}{2}\left(\frac{\alpha(x)}{1+\sqrt{1+\alpha(x)}}\right)^2, \quad |x| \ge \tilde{z}_0. \tag{3.35}$$

In the following estimate of $|\varepsilon(x)|$, we use (3.35), (3.32), (3.30) and (1.10):

$$\begin{aligned}|\varepsilon(x)| &\le \frac{|\alpha(x)|}{2} + \frac{|\alpha(x)|^2}{2} = |\alpha(x)|\frac{|1+|\alpha(x)|}{2} \le \frac{1+0.0332}{2}|\alpha(x)| = 0.5166|\alpha(x)| \\ &= 0.5166|\delta_4(x)|\frac{\rho(x)}{2d(x)} \le 44.23 \cdot \frac{3}{4} \cdot 0.5166h(x) < 18h(x), \qquad |x| \ge z_0.\end{aligned} \tag{3.36}$$

Lemma 3.1 now follows from (3.33), (3.35) and (3.36). □

4. Proof of the main result

In this section we finish the proof of formula (1.19). Note that this part more or less coincides with the corresponding fragment of [3] and is reproduced here, with minor changes, only for the sake of completeness.

Lemma 4.1. *For $x \in R$, we have the inequality*

$$|\rho'(x)| \le |\varkappa(x) - 1|, \qquad \varkappa(x) \overset{def}{=} \frac{v'(x)}{v(x)} \cdot \frac{u(x)}{|u'(x)|}. \tag{4.1}$$

Proof. From (1.3), (1.4) and (3.9), it follows that

$$|\rho'(x)| = |v'(x)u(x) + v(x)u'(x)| = |u'(x)|v(x)|\varkappa(x) - 1| < |\varkappa(x) - 1|.$$

□

Lemma 4.2. *For $x \in R$, the formula*

$$y(t) = v'(x)u(t) - u'(x)v(t), \qquad t \in R \tag{4.2}$$

determines the solution of the Cauchy problem

$$y''(t) = q(t)y(t), \qquad t \in R, \tag{4.3}$$

$$y(t)\,\Big|_{t=x}=1, \qquad y'(t)\,\Big|_{t=x}=0. \tag{4.4}$$

In addition, the following inequalities hold:

$$y'(t)\le 0 \qquad \text{for}\quad t\le x; \qquad y'(t)\ge 0 \qquad \text{for}\quad t\ge x. \tag{4.5}$$

Proof. Let us check (4.5) for $t\ge x$. (The other assertions of the lemma immediately follows from the properties of the PFSS $\{u(x),v(x)\}$ of equation (1.1) (see (1.4).) Let us show that $y(t)>0$ for $t>x$. If this is not the case, let x_0 be the smallest positive root of the equation $y(t)=0$ ($x_0>0$ because of (4.4)). Then $y'(x_0)\le 0$. Indeed, if $y'(x_0)>0$, then $y(t)<0$ for $t<x_0$ because $y(x_0)=0$. But then (4.4) implies that the equation $y(t)=0$ has a root in the interval $(0,x_0)$ which contradicts the definition of x_0. Thus $y'(x_0)\le 0$. On the other hand, from (4.3) it follows that

$$y'(x_0)=\int_0^{x_0} q(\xi)y(\xi)d\xi\ge 0 \quad\Rightarrow\quad y'(x_0)=0.$$

Hence $y(t)=0$ because $y(x_0)=y'(x_0)=0$. Contradiction.

Since $y(t)>0$ for $\ge x$, according to (1.2) and (4.3)–(4.4) we get

$$y'(t)=\int_0^t q(\xi)y(\xi)d\xi\ge 0 \qquad \text{for}\quad t\ge x.$$

The case $t\le x$ is treated in a similar way. □

Let $q(x)\in H$. Let us introduce the functions (see (2.14))

$$\tilde u(t)=y(t)\int_t^{\omega^+(x)}\frac{d\xi}{y(\xi)^2},\quad \tilde v(t)=y(t)\int_{\omega^-(x)}^t\frac{d\xi}{y(\xi)^2},\quad t\in\omega(x),\quad x\in R. \tag{4.6}$$

In (4.6), we assume that $y(t)$ is the solution of the problem (4.3)–(4.4).

Lemma 4.3. *The functions* (4.6) *are solutions of equation* (4.3) *and satisfy the relations*

$$\tilde u(\omega^+(x))=\tilde v(\omega^-(x))=0,\quad \tilde u(t)\ge 0,\quad \tilde v(t)\ge 0 \qquad \text{for}\quad t\in\omega(x), \tag{4.7}$$

$$\tilde v'(t)\tilde u(t)-\tilde u'(t)\tilde v(t)=\int_{\omega(x)}\frac{d\xi}{y(\xi)^2},\quad t\in\omega(x). \tag{4.8}$$

Proof. The relations (4.7) are obvious. Equality (4.8) is checked by a straightforward calculation. □

Lemma 4.4. *For $x\in R$, we have the equalities*

$$\frac{v(x)}{v'(x)}=\tilde v(x)\left[1-\frac{v(\omega^-(x))}{v(x)}\,\frac{1}{y(\omega^-(x))}\right]^{-1}, \tag{4.9}$$

$$\frac{u(x)}{|u'(x)|}=\tilde u(x)\left[1-\frac{u(\omega^+(x))}{u(x)}\ \frac{1}{y(\omega^+(x))}\right]^{-1}. \tag{4.10}$$

Here $y(\cdot)$ is the solution of problem (4.3)–(4.4).

Proof. The equalities (4.9)–(4.10) follow from (1.9), (4.2) and (4.6). For example,

$$\tilde v(x)=\int_{\omega^-(x)}^{x}\frac{d\xi}{[v'(x)u(\xi)-u'(x)v(\xi)]^2}=\int_{\omega^-(x)}^{x}\frac{1}{\rho(\xi)}\ \frac{\exp\left(-\int_{x_0}^{\xi}\frac{ds}{\rho(s)}\right)d\xi}{\left[v'(x)\exp\left(-\int_{x_0}^{\xi}\frac{ds}{\rho(s)}\right)-u'(x)\right]^2}$$

$$=\int_{\omega^-}^{x}\frac{1}{v'(x)}d\left[v'(x)\exp\left(-\int_{x_0}^{\xi}\frac{ds}{\rho(s)}\right)-u'(x)\right]^{-1}$$

$$=\frac{1}{v'(x)}\left[v'(x)\exp\left(-\int_{x_0}^{\xi}\frac{ds}{\rho(s)}\right)-u'(x)\right]^{-1}\Big|_{\omega^-(x)}^{x}$$

$$=\frac{1}{v'(x)}\ \frac{v(\xi)}{v'(x)u(\xi)-u'(x)v(\xi)}\Big|_{\omega^-(x)}^{x}=\frac{1}{v'(x)}\ \frac{v(\xi)}{y(\xi)}\Big|_{\omega^-(x)}^{x}=\frac{1}{v'(x)}\left[v(x)-\frac{v(\omega^-(x))}{y(\omega^-(x))}\right]$$

$$=\frac{v(x)}{v'(x)}\left[1-\frac{v(\omega^-(x))}{v(x)}\ \frac{1}{y(\omega^-(x))}\right]\ \Rightarrow\ (4.9).$$

The equality (4.10) is checked in a similar way. □

Lemma 4.5. *Suppose $q(x)\in H$. Then for $x\in R$ we have the equality (see* (2.15)*):*

$$\varkappa(x)=\frac{v'(x)}{v(x)}\ \frac{u(x)}{|u'(x)|}=\frac{\tilde u(x)}{\tilde v(x)}(1+\nu(x)),\quad |\nu(x)|\le\sqrt{6c_3}\exp\left(-\frac{\sqrt{k(x)}}{3c_3}\right). \tag{4.11}$$

Proof. By Lemma 4.4, to prove (4.11) it is enough to show that for $x\in R$, the following inequality holds:

$$\max\left\{\frac{v(\omega^-(x))}{v(x)}\ \frac{1}{y(\omega^-(x))},\quad \frac{u(\omega^+(x))}{u(x)}\ \frac{1}{y(\omega^+(x))}\right\}\le\sqrt{6c_3}\exp\left(-\frac{\sqrt{k(x)}}{3c_3}\right). \tag{4.12}$$

By (4.5), (1.9), (1.10) and (2.15), we have

$$\frac{u(\omega^+(x))}{u(x)}\ \frac{1}{y(\omega^+(x))}\le\frac{\omega^+(x))}{u(x)}$$

$$=\sqrt{\frac{\rho(\omega^+(x))}{d(\omega^+(x))}\cdot\frac{d(\omega^+(x))}{d(x)}\cdot\frac{d(x)}{\rho(x)}}\exp\left(-\frac12\int_x^{\omega^+(x)}\frac{d(\xi)}{\rho(\xi)}\cdot\frac{d(x)}{d(\xi)}\cdot\frac{d\xi}{d(x)}\right)$$

$$\le\sqrt{\frac32\cdot c_3\cdot 4}\exp\left(-\frac12\int_x^{\omega_+(x)}\frac23\ \frac{1}{c_3}\ \frac{d\xi}{d(x)}\right)=\sqrt{6c_3}\exp\left(-\frac{\sqrt{k(x)}}{3c_3}\right).$$

The second inequality of (4.12) is checked in a similar way. □

To study $\tilde u(x)/\tilde v(x)$, let us look at the solution $y(t)$ of problem (4.3)–(4.4) more closely than in Lemma 4.2. Suppose $q(x)\in H$. Denote

$$\chi(x)=[0,\sqrt{k(x)}d(x)],\qquad x\in R. \tag{4.13}$$

In problem (4.3)–(4.4), we change variables:

$$y_1(x)=y(x-z),\qquad z\in\chi(x), \tag{4.14}$$

$$y_2(x)=y(x+z),\qquad z\in\chi(x). \tag{4.15}$$

It is easy to see that $y_1(z)$ and $y_2(z)$ are solutions of the following Cauchy problems, respectively:

$$y_1''=q(x-z)y_1(x),\qquad y_1(0)=1,\qquad y_1'(0)=0, \tag{4.16}$$

$$y_2''=q(x+z)y_1(x),\qquad y_2(0)=1,\qquad y_2'(0)=0. \tag{4.17}$$

Lemma 4.6. *Suppose $q(x)\in H$, and let t_0 be a positive number such that $k(x)\ge 64c_2^2$ for all $|x|\ge t_0$ (see* (1.12), (1.14)*). Then for $|x|\ge t_0$ and $z\in\chi(x)$ (see* (4.13)*), the following relations hold:*

$$\frac{y_2(z)}{y_1(z)}=\frac{y(x+z)}{y(x-z)}=1+\gamma(z),\qquad |\gamma(z)|\le 1.2F(x)\le\frac{1.2c_2}{\sqrt{k(x)}}. \tag{4.18}$$

Proof. Let us introduce some notation:

$$\beta(z)=\frac{y_2(z)}{y_1(z)},\quad \varphi(z)=\int_0^z(q(x+\xi)-q(x-\xi))d\xi,\quad \psi(z)=\max_{t\in[0,z]}|\varphi(t)|,\quad z\in\chi(x). \tag{4.19}$$

By (4.5), we get $y_1'(z)\ge 0$, $y_2'(z)\ge0$ for $z\ge0$. Therefore, we also have $[y_1(z)\cdot y_2(z)]'\ge0$ for $z\ge0$. Integrating by parts, we get

$$\begin{aligned}|\beta'(z)|&=\left|\frac{d}{dz}\left(\frac{y_2(z)}{y_1(z)}\right)\right|=\frac{1}{y_1(z)^2}\left|\int_0^z(q(x+\xi)-q(x-\xi))y_1(\xi)y_2(\xi)d\xi\right|\\
&\le\frac{y_2(z)}{y_1(z)}|\varphi(z)|+\frac{1}{y_1(z)^2}\left|\int_0^z\varphi(\xi)[y_1(\xi)y_2(\xi)]'d\xi\right|\\
&\le\frac{y_2(z)}{y_1(z)}\psi(z)+\frac{\psi(z)}{y_1(z)^2}\int_0^z|[y_1(\xi)y_2(\xi)]'|d\xi\\
&=\frac{y_2(z)}{y_1(z)}\psi(z)+\frac{\psi(z)}{y_1(z)^2}(y_1(z)y_2(z)-1)\\
&\le 2\psi(z)\frac{y_2(z)}{y_1(z)}=2\psi(z)\beta(z),\quad z\in\chi(x).\end{aligned} \tag{4.20}$$

Since for $z \in \chi(x)$ the function $\psi(x)$ satisfies the inequalities (see (1.14))

$$\psi(z) = \sup_{t\in[0,z]} \left| \int_0^t (q(x+\xi) - q(x-\xi))d\xi \right| \le \sup_{t\in\chi(x)} \left| \int_0^t (q(x+\xi) - q(x-\xi))d\xi \right|$$
$$\le \frac{F(x)}{\sqrt{k(x)}d(x)}, \tag{4.21}$$

by (4.21) we can continue estimate (4.20):

$$|\beta'(z)| \le \frac{2F(x)}{\sqrt{k(x)}d(x)}\beta(z), \quad z \in \chi(x), \quad x \in R$$
$$\Rightarrow \ -\frac{2F(x)}{\sqrt{k(x)}d(x)} \le \frac{\beta'(z)}{\beta(z)} \le \frac{2F(x)}{\sqrt{k(x)}d(x)}, \quad z \in \chi(x), \quad x \in R. \tag{4.22}$$

Since $\beta(0) = 1$, from (4.22) we get

$$\exp(-2F(x)) \le \beta(z) \le \exp(2F(x)), \quad z \in \chi(x), \quad x \in R. \tag{4.23}$$

Let us check that $F(x) \to 0$ as $|x| \to \infty$. According to (1.14) and (1.15), we have

$$F(x) = \sqrt{k(x)}d(x) \sup_{z\in\chi(x)} \left| \int_0^z (q(x+t) - q(x-t)dt \right|$$
$$\le \frac{1}{\sqrt{k(x)}} \left[k(x)d(x) \sup_{z\in[0,k(x)d(x)]} \left| \int_0^z (q(x+t) - q(x-t))dt \right| \right] \le \frac{c_2}{\sqrt{k(x)}}, \quad x \in R \tag{4.24}$$

$$\Rightarrow \ F(x) \to 0 \qquad \text{as} \quad |x| \to \infty. \tag{4.25}$$

Let now $|x| \ge t_0$. Then from the assumption of the lemma and (4.24), it follows that

$$\alpha(x) \overset{\text{def}}{=} 2F(x) \le \frac{2c_2}{\sqrt{k(x)}} \le \frac{1}{4}, \quad |x| \ge t_0. \tag{4.26}$$

From (4.26), we get

$$e^{\alpha x} = 1 + \alpha(x) + \sum_{n=2}^{\infty} \frac{(\alpha(x))^n}{n!} \le 1 + \alpha(x) + \frac{\alpha(x)^2}{2} \sum_{k=0}^{\infty} \left(\frac{\alpha(x)}{2} \right)^k$$
$$= 1 + \alpha(x) + \frac{\alpha(x)^2}{2-\alpha(x)} \le 1 + \alpha(x) + \frac{4}{7}(\alpha(x))^2 \le 1 + \alpha(x) + \frac{\alpha(x)}{7} \le 1 + 1.2\alpha(x). \tag{4.27}$$

The lemma follows from (4.26), (4.27) and (4.23). $\square$

Lemma 4.7. *Suppose $q(x) \in H$, let $y(t)$ be the solution of problem* (4.3)–(4.4)*, and let t_0 be the number from Lemma 4.6. Then for $|x| \ge t_0$, the following relations hold (see* (2.14)*):*

$$\int_x^{\omega^+(x)} \frac{dt}{y(t)^2} = (1 + \tau(x)) \int_{\omega^-(x)}^x \frac{dt}{y(t)^2}, \quad |\tau(x)| \le 3.6F(x) \le \frac{3.6c_2}{\sqrt{k(x)}}. \tag{4.28}$$

Proof. By the definition of t_0, for $|x| \geq t_0$ and $z \in \chi(x)$, we have the following estimate for $|\gamma(z)|$ (see (4.18)):

$$|\gamma(z)| \leq 1.2F(x) \leq \frac{1.2c_2}{\sqrt{k(x)}} \leq 1.2 \cdot \frac{1}{8} = 0.15. \tag{4.29}$$

From (4.18) and (4.29), we get

$$\begin{aligned}
\int_x^{\omega^+(x)} \frac{dt}{y(t)^2} &= \int_0^{\sqrt{k(x)}dx} \frac{dz}{y(x+z)^2} = \int_0^{\sqrt{k(x)}d(x)} \frac{dz}{(1+\gamma(z))^2 y(x-z)^2} \\
&= \int_0^{\sqrt{k(x)}d(x)} \frac{dz}{y(x-z)^2} - \int_0^{\sqrt{k(x)}d(x)} \gamma(z) \frac{2+\gamma(z)}{(1+\gamma(z))^2} \frac{dz}{y(x-z)^2} \\
&= \left[1 - \int_0^{\sqrt{k(x)}d(x)} \gamma(z) \frac{2+\gamma(z)}{(1+\gamma(z))^2} \frac{dz}{y(x-z)^2} \left(\int_0^{\sqrt{k(x)}d(x)} \frac{dz}{y(x-z)^2}\right)^{-1}\right] \\
&\quad \cdot \int_0^{\sqrt{k(x)}d(x)} \frac{dz}{y(x-z)^2} \\
&\overset{\text{def}}{=} (1+\tau(x)) \int_0^{\sqrt{k(x)}d(x)} \frac{dt}{y(x-z)^2} = (1+\tau(x)) \int_{\omega^-(x)}^x \frac{dt}{y(t)^2}, \quad |x| \geq t_0.
\end{aligned} \tag{4.30}$$

It remains to prove the estimate $|\tau(x)|$ from (4.28). We use relations (4.29) and (4.18):

$$|\tau(x)| \leq \max_{z \in \chi(x)} |\gamma(x)| \frac{|2+\gamma(z)|}{(1+\gamma(z))^2} \leq 1.2F(x) \frac{2.15}{0.85^2} \leq 3.6F(x) \leq \frac{3.6c_2}{\sqrt{k(x)}}.$$

□

Lemma 4.8. *Suppose $q(x) \in H$, and let t_0 be the number from Lemma 4.6. Then for $|x| \geq t_0$, the following inequalities hold:*

$$|\rho'(x)| \leq 3.6 \left[F(x) + \sqrt{c_3} \exp\left(-\frac{\sqrt{k(x)}}{3c_3}\right)\right] \leq 11 \frac{c_2 + c_3\sqrt{c_3}}{k(x)}, \quad |x| \geq t_0. \tag{4.31}$$

Proof. Below when estimating $\varkappa(x)$, we use (4.11), (4.6) and (4.28):

$$\begin{aligned}
\varkappa(x) &= \frac{v'(x)}{v(x)} \frac{u(x)}{|u'(x)|} = \frac{\tilde{u}(x)}{\tilde{v}(x)}(1+\nu(x)) = \int_x^{\omega^+(x)} \frac{dt}{y(t)^2} \left(\int_{\omega^-(x)}^x \frac{dt}{y(t)^2}\right)^{-1} (1+\nu(x)) \\
&= (1+\tau(x))(1+\nu(x)) = 1 + \tau(x) + \nu(x) + \tau(x)\nu(x) \overset{\text{def}}{=} 1 + \mu(x), \quad |x| \geq t_0.
\end{aligned} \tag{4.32}$$

From (4.28) and (4.26), we get

$$|\tau(x)| \leq 3.6 \frac{c_2}{\sqrt{k(x)}} \leq 0.45, \qquad |x| \geq t_0. \tag{4.33}$$

Inequality (4.33), together with (4.32), (4.28) and (4.11), lead to the estimates:

$$
\begin{aligned}
|\mu(x)| &\le |\tau(x)| + |\nu(x)| + |\tau(x)|\ |\nu(x)| \le |\tau(x)| + 1.45|\nu(x)| \\
&\le 3.6F(x) + 3.6\sqrt{c_3}\exp\left(-\frac{\sqrt{k(x)}}{3c_3}\right) \le 4F(x) + \frac{11c_3^{3/2}}{\sqrt{k(x)}} \\
&\le 11\frac{c_2 + c_3\sqrt{c_3}}{\sqrt{k(x)}}, \quad |x| \ge t_0. \qquad (4.34)
\end{aligned}
$$

Inequalities (4.31) follow from (4.34) and (4.1). □

Proof of Theorems 1.4 and 1.5. Suppose $q(x) \in H$. Then (4.31) implies (1.16) for $|x| \gg 1$. In addition, (4.31) and (1.12) lead to (3.1). Hence by Lemma 3.1 we get (3.2). We estimate $|\epsilon(x)|$ in (3.3) using the estimate for $h(x)$. Below for $|x| \gg 1$ we use (4.31) (4.31), (1.13) and (4.24):

$$
\begin{aligned}
h(x) &= \sup_{t\in\Delta(x)} |\rho'(t)| \le 3.6 \sup_{t\in\Delta(x)} F(x) + 3.6\sqrt{c_3} \sup_{t\in\Delta(x)} \left(-\frac{\sqrt{k(t)}}{3c_3}\right) \\
&\le 3.6 \sup_{t\in\Delta(x)} F(t) + 3.6\sqrt{c_3}\exp\left(-(3c_3c_1^{1/2})^{-1}\sqrt{k(x)}\right) \\
&\le c\left\{\sup_{t\in\Delta(x)} F(t) + \exp\left(-c^{-1}\sqrt{k(x)}\right)\right\} = c\beta(x). \qquad (4.35)
\end{aligned}
$$

From (4.35) and (3.3), we get (1.19). From (4.24) and (1.13), we obtain

$$
\begin{aligned}
\beta(x) &= \sup_{t\in\Delta(x)} F(t) + \exp\left(-c^{-1}\sqrt{k(x)}\right) \le \sup_{t\in\Delta(x)} \frac{c_2}{\sqrt{k(t)}} + \frac{c}{\sqrt{k(x)}} \\
&\le \frac{c_2\sqrt{c_1}}{\sqrt{k(x)}} + \frac{c}{\sqrt{k(x)}} \le \frac{c}{\sqrt{k(x)}} \quad \Rightarrow \quad (1.20).
\end{aligned}
$$

Thus Theorem 1.4 is proved.

To prove Theorem 1.5, we set $|x| \ge s_1 > s_0$ (see (1.26) – (1.27)). Then $s_0 > t_0$ because of (1.26), where t_0 is the number from Lemma 4.6. Hence $|\rho'(x)| \le 10^{-3}$ according to (4.31) and (1.26). Formula (3.2) is proved similarly to Theorem 1.4. Since (4.31) coincides with (1.27), it remains to estimate $|\varepsilon(x)|$ using (3.6). We use one of the inequalities (4.35) and obtain:

$$
|\varepsilon(x)| \le 18h(x) \le 65\left(\sup_{t\in\Delta(x)} F(t) + \sqrt{c_3}\exp\left(-(3c_3\sqrt{c_1})^{-1}\sqrt{k(x)}\right)\right) = \eta_2(x).
$$

Theorem 1.5 is proved. □

5. Comparison of two asymptotic formulas

In this section we prove Theorem 1.7.

Proof of Theorem 1.7. Suppose $q(x) \in H$ and $q(x) \ge 1$ for $x \in R$. Consider (1.17) and (1.19). In these relations $\alpha(x)$ and $\beta(x)$ are positive, continuous for $x \in R$ functions, $\beta(x) \le \alpha(x)$ for $x \in R$ and $\alpha(x) \to 0$, $\beta(x) \to 0$ as $|x| \to \infty$. Therefore the theorem will be proved if (see (1.31))

$$L = \sup_{q(\cdot)\in H} \overline{\lim_{|x|\to\infty}} \frac{\alpha(x)}{\beta(x)} = \infty. \tag{5.1}$$

Note that to prove (5.1), it suffices to give an example of the function $q(\cdot)$ which, on the one hand, satisfies the above-mentioned assumptions and, on the other hand, for the functions $\alpha(x)$ and $\beta(x)$ constructed by $q(\cdot)$ the following equality holds:

$$\tilde L = \overline{\lim_{|x|\to\infty}} \frac{\alpha(x)}{\beta(x)} = \infty. \tag{5.2}$$

(Indeed, if (5.2) holds, then $\infty = \tilde L \le L \le \infty \ \Rightarrow\ \tilde L = L = \infty$.)

Let us construct such a function. Denote

$$\sigma_n = \left[\sigma_n^{(-)}, \sigma_n^{(+)}\right) = [n^2, (n+1)^2), \quad q_n = \left(1 + \frac{1}{n}\right)^n, \quad n = 1, 2, \dots \tag{5.3}$$

Suppose $q(-x) = q(x)$ for $x \ge 0$ and

$$q(x) = \begin{cases} q_n & \text{if} \quad x \in \sigma_n, \quad n = 1, 2, \dots \\ 2 & \text{if} \quad x \in [0, 1). \end{cases} \tag{5.4}$$

Clearly, we have $1 \le q(x) \in L_1^{\mathrm{loc}}(R)$, $x \in R$. We show that $q(x) \in H$. We need to estimate the function $d(x)$. Since

$$2 \le q(x) \le 3 \qquad \text{for} \quad x \in R, \tag{5.5}$$

by (1.11) we have

$$2 = d(x) \int_{x-d(x)}^{x+d(x)} q(t)dt \ge d(x) \int_{x-d(x)}^{x+d(x)} 2dt = 4d(x)^2 \ \Rightarrow\ d(x) \le \frac{1}{\sqrt{2}}, \ x \in R,$$

$$2 = d(x) \int_{x-d(x)}^{x+d(x)} q(t)dt \le d(x) \int_{x-d(x)}^{x+d(x)} 3dt = 6d(x) \ \Rightarrow\ d(x) \le \frac{1}{\sqrt{3}}, \ x \in R.$$

Hence

$$\frac{1}{\sqrt{3}} \le d(x) \le \frac{1}{\sqrt{2}}, \qquad x \in R. \tag{5.6}$$

We introduce the function

$$k(x) = \begin{cases} \sqrt{|x|}, & \text{if} \quad |x| \ge 4 \\ 2, & \text{if} \quad |x| \le 4 \end{cases} \tag{5.7}$$

Let us check that in case (5.7) all the assumptions of Definition 1.2 are satisfied. From (5.6) and (5.7), we get relations (1.12) and (1.13). In particular, (1.12) immediately follows from (5.7). We prove (1.13). Let $x \geq 9$. Then

$$\begin{aligned} & x-\sqrt{x}=x\left(1-\frac{1}{\sqrt{x}}\right)\geq 9\left(1-\frac{1}{3}\right)=6\geq 4 \\ \Rightarrow\quad & [x-k(x)d(x),x+k(x)d(x)]\subseteq\left[x-\sqrt{x},x+\sqrt{x}\right], \\ & \left[x-\sqrt{x},x+\sqrt{x}\right]\cap[-4,4]=\emptyset. \end{aligned}$$

Thus for $x \geq 9$, inequalities (1.13) are true.

The estimates proved for $|x| \geq 9$ can be easily extended to the whole number axis using Lemma 2.6. It remains to check (1.14). Consider $\Phi(x)$ (see (1.14)) for $x \in \sigma_n$, $n \geq 2$. Clearly, if $x \in \sigma_n$, then

$$\left.\begin{array}{l} x+k(x)d(x)\leq x+\sqrt{x}\leq (n+1)^2+n+1<(n+2)^2 \\ x-k(x)d(x)\geq x-\sqrt{x}\geq n^2-n>(n-1)^2; \end{array}\right\}\Rightarrow$$

$$[x-k(x)d(x),x+k(x)d(x)]\subseteq\sigma_{n-1}\cup\sigma_n\cup\sigma_{n+1}\quad\text{for}\quad x\in\sigma_n,\quad n\geq 2. \tag{5.8}$$

Now from the condition $x \in \sigma_n$ and (5.8), (5.4), (5.7), (5.6) and (1.14), we get

$$\begin{aligned} \Phi(x) &= k(x)d(x)\sup_{x\in[0,k(x)d(x)]}\left|\int_0^z (q(x+t)-q(x-t)dt\right| \\ &\leq \frac{n+1}{\sqrt{2}}\max\left\{|q_{n+1}-q_n|,|q_n-q_{n-1}|\right\}\cdot\frac{n+1}{\sqrt{2}} \\ &= \frac{(n+1)^2}{2}\max\left\{|q_{n+1}-q_n|,|q_n-q_{n-1}|\right\}. \end{aligned} \tag{5.9}$$

Since the following inequalities hold (see [10, Section I, problem 170]):

$$\left.\begin{array}{l} \frac{e}{2n+2}<e-q_n<\frac{e}{2n+1} \\ -\frac{e}{2n+3}<q_{n+1}-e<-\frac{e}{2n+4} \end{array}\right\}\Rightarrow\frac{e}{(2n+2)(2n+3)}<q_{n+1}-q_n<\frac{3e}{(2n+1)(2n+4)} \tag{5.10}$$

by (5.9) and (5.10), we obtain

$$\Phi(x)\leq\frac{(n+1)^2}{2}\,\frac{3e}{(2n-1)(2n+2)}\leq C<\infty\qquad\text{for}\quad x\in\sigma_n,\quad n\geq 2. \tag{5.11}$$

We omit the obvious proof of (1.14) using (5.11). Thus $q(x) \in H$, and it remains to prove (5.2). Let (see (5.3))

$$x_n=\frac{\sigma_n^{(-)}+\sigma_n^{(+)}}{2}=n^2+n+\frac{1}{2},\qquad n\geq 1. \tag{5.12}$$

Let us compute $\sup_{t\in\Delta(x_n)} F(t)$. Note that if $t\in\Delta(x_n)$, then (5.6) implies elementary inequalities

$$\left.\begin{aligned} &t+\sqrt{k(x)}d(t)\le x_n+\tfrac{1}{\sqrt2}+\sqrt{k\left(x_n+\tfrac{1}{\sqrt2}\right)}\tfrac{1}{\sqrt2}<(n+1)^2 \quad \text{for} \quad n\gg1,\\ &t-\sqrt{k(x)}d(t)\ge x_n-\tfrac{1}{\sqrt2}-\sqrt{k\left(x_n-\tfrac{1}{\sqrt2}\right)}\tfrac{1}{\sqrt2}>n^2 \quad \text{for} \quad n\gg1,\end{aligned}\right\}\Rightarrow$$

$$\left[t-\sqrt{k(x)}d(t),t+\sqrt{k(x)}d(t)\right]\subset\sigma_n \quad \text{for} \quad t\in\Delta_n \quad \text{and} \quad n\gg1. \tag{5.13}$$

Furthermore, according to (5.13) and (5.4), we have

$$q(t+\xi)=q(t-\xi)=q_n \qquad \text{for} \qquad |\xi|\le\sqrt{k(t)}d(t), \quad t\in\Delta(x_n), \quad n\gg1,$$

and therefore $\sup_{t\in\Delta(x_n)} F(t)=0$ (see (1.15)). By (1.20), this implies

$$\beta(x_n)=\exp\left(-c^{-1}\sqrt{k(x_n)}\right), \qquad n\gg1. \tag{5.14}$$

Now we consider the value

$$\tilde F(s)\,\big|_{s=\sigma_n^{(+)}}=\sqrt{k(x)}d(s)\left|\int_0^{\sqrt{k(s)}ds}(q(s+t)-q(s-t))dt\right|\,\Bigg|_{s=\sigma_n^{(+)}}.$$

Since for $t\in\left(0,\sqrt{k\left(\sigma_n^{(+)}\right)}d\left(\sigma_n^{(+)}\right)\right]$, we have

$$q(\sigma_n^{(+)}+t)=q_{n+1}, \qquad q(\sigma_n^{(+)}-t))=q_n,$$

using (5.10), (5.6) and (5.7), we get

$$\tilde F(\sigma_n^{(+)})=k(\sigma_n^{(+)})d(\sigma_n^{(+)})^2(q_{n+1}-q_n)\ge\frac{n+1}{3}\,\frac{e}{(2n+2)(2n+3)}=\frac{e}{6(2n+3)}.$$

The last inequality yields the estimates

$$\sup_{t\ge x_n-d(x_n)}F(t)\ge F(\sigma_n^{(+)})\ge\tilde F(\sigma_n^{(+)})\ge\frac{e}{6(2n+3)}, \qquad n\gg1. \tag{5.15}$$

From (5.15) and (1.18), we finally get

$$\alpha(x_n)\ge\exp\left(-c^{-1}\sqrt{k(x_n)}\right)+\frac{e}{6(2n+3)}, \qquad n\gg1. \tag{5.16}$$

Relations (5.14) and (5.15) imply (5.1). Indeed,

$$\frac{\alpha(x_n)}{\beta(x_n)}\ge1+\frac{e}{6(2n+3)}\exp\left(c^{-1}\sqrt{k(x_n)}\right) \qquad \text{for} \quad n\gg1$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{\alpha(x_n)}{\beta(x_n)}=\infty \quad \Rightarrow \quad \tilde L=\infty \quad \Rightarrow \quad L=\infty.$$

$\square$

6. Properties of solutions of the Riccati equation

In this section, we prove Theorem 1.8. Below we use the following assertion.

Theorem 6.1. [5, §402] *The general solution of equation* (1.32) *is of the form*

$$y(x)=\frac{c_1v'(x)+c_2u'(x)}{c_1v(x)+c_2u(x)}. \tag{6.1}$$

Here $\{u(x),v(x)\}$ is a PFSS of equation (1.1), c_1,c_2 are arbitrary constant, $|c_1|+|c_2|\neq 0$.

Proof of Theorem 1.8. Let $q(x)\in H$. Set

$$y_2(x)=\frac{v'(x)}{v(x)},\qquad y_1(x)=\frac{u'(x)}{u(x)},\qquad x\in R. \tag{6.2}$$

Then by (3.8) and Theorem 1.4, we get (1.33):

$$\lim_{|x|\to\infty} y_2(x)d(x)=\lim_{|x|\to\infty}\frac{v'(x)}{v(x)}d(x)=\lim_{|x|\to\infty}(1+\rho'(x))\frac{d(x)}{2\rho(x)}=1,$$

$$\lim_{|x|\to\infty} y_1(x)d(x)=\lim_{|x|\to\infty}\frac{u'(x)}{u(x)}d(x)=\lim_{|x|\to\infty}(\rho'(x)-1)\frac{d(x)}{2\rho(x)}=-1.$$

Consider the second part of assertion A). Suppose that there exists a solution $y(x)$ of equation (1.33) which satisfies the following properties of the solution $y_2(x)$:

1) the solution $y(x)$ is defined for all $x\in R$;
2) the following equalities hold:

$$\lim_{x\to-\infty} y(x)d(x)=\lim_{x\to\infty} y(x)d(x)=1. \tag{6.3}$$

Let us show that 1) and 2) imply $y(x)\equiv y_2(x)$ for $x\in R$.

We need the following assertion.

Lemma 6.2. *Suppose that conditions* (1.2) *hold, and let* $y(x)$ *be a solution of equation* (1.32) *such that* $y(x)\neq y_1(x)$, $y(x)\neq y_2(x)$. *Then if the solution* $y(x)$ *is defined for all* $x\in R$, *then*

$$y_1(x)<y(x)<y_2(x)\qquad \text{for}\qquad x\in R. \tag{6.4}$$

Proof. Suppose that $y(x_0)>y_2(x_0)$ for some $x_0\in R$. By the hypothesis of the lemma, in representation (6.1) we have $c_1\neq 0$, $c_2\neq 0$, and therefore

$$y(x)=\frac{v'(x)+\theta u'(x)}{v(x)+\theta u(x)},\qquad \theta\neq 0,\quad \theta=\frac{c_2}{c_1},\quad x\in R. \tag{6.5}$$

Since $y(x_0) > y_2(x_0)$, (1.3) and (1.4) imply

$$0 < y(x_0) - y_2(x_0) = \frac{v'(x_0) + \theta u'(x_0)}{v(x_0) + \theta(x_0)} - \frac{v'(x_0)}{v(x_0)} = -\frac{\theta}{(v(x_0 + \theta u(x_0))v(x_0)}$$

$$= -\frac{1}{\theta^{-1} + u(x_0)/v(x_0)} \frac{1}{v(x_0)^2} \quad \Rightarrow \quad \theta^{-1} + \frac{u(x_0)}{v(x_0)} < 0 \quad \Rightarrow \quad \theta < 0.$$

Let $\varphi(x) = u(x)/v(x)$, $x \in R$. According to (1.3), (1.4) and (1.5), this function satisfies the properties $\varphi(x) \to \infty$ as $x \to -\infty$, $\varphi(x) \to 0$ as $x \to \infty$

$$\varphi'(x) = \frac{u'(x)v(x) - v'(x)u(x)}{v(x)^2} = -\frac{1}{v(x)^2} < 0 \qquad \text{for} \quad x \in R.$$

Hence there exists x_1 such that $\varphi(x_1) = -\theta^{-1}$, or, equivalently,

$$v(x_1) + \theta(x_1) = 0. \tag{6.6}$$

Together with equality (6.6), the following inequality holds:

$$v'(x_1) + \theta u'(x_1) = v'(x_1) + |\theta u'(x_1)| > 0 \tag{6.7}$$

(see (3.9)). From (6.6), (6.7) and (6.5), it follows that the solution $y(x)$ is not defined for $x = x_1$; contradiction. Hence $y(x) \le y_2(x)$ for all $x \in R$. But $y(x) \ne y_2(x)$ by hypothesis which leads to the upper estimate in (6.4). The second inequality of (6.4) can be checked in a similar way. □

Corollary 6.3. *Assuming the hypothesis of Lemma 6.2, the solution of equation* (1.32) *is of the form* (6.5) *with* $\theta > 0$.

Proof. Taking into account all that was mentioned above, it only remains to check that $\theta > 0$. From (6.4), (1.3) and (1.4), it follows that

$$0 < y(x) - y_1(x) = \frac{v'(x) + \theta u'(x)}{v(x) + \theta u(x)} - \frac{u'(x)}{u(x)} = \frac{1}{u(x)(v(x) + \theta u(x))}, \qquad x \in R;$$

$$0 < y_2(x) - y(x) = \frac{v'(x)}{v(x)} - \frac{v'(x) + \theta u'(x)}{v(x) + \theta u(x)} = \frac{\theta}{(v(x)(v(x) + \theta u(x))}.$$

The first inequality implies $v(x) + \theta u(x) > 0$, $x \in R$. Then $\theta > 0$ in view of the second inequality. □

We can now finish the proof of assertion A). First note that if $q(x) \in H$, then in addition to (1.5) we have the relations

$$\lim_{x \to -\infty} \frac{v'(x)}{u'(x)} = \lim_{x \to \infty} \frac{u'(x)}{v'(x)} = 0. \tag{6.8}$$

Indeed, from (3.9), (3.8), (1.16), (1.12) and (1.5), it follows that

$$\lim_{x\to-\infty}\frac{v'(x)}{u'(x)}=\lim_{x\to-\infty}\frac{v'(x)}{v(x)}\,\frac{v(x)}{u(x)}\,\frac{u(x)}{u'(x)}=\lim_{x\to-\infty}\frac{1+\rho'(x)}{2\rho(x)}\,\frac{v(x)}{u(x)}\,\frac{2\rho(x)}{\rho'(x)-1}$$
$$=\lim_{x\to-\infty}\frac{1+\rho'(x)}{\rho'(x)-1}\lim_{x\to-\infty}\frac{v(x)}{u(x)}=0.$$

The second equality of (6.8) can be proved in a similar way. Let $y(x)$ be a solution of (1.32) which does not coincide with $y_2(x)$ for $x\in R$ and satisfies properties 1)–2) (see above). Then by Corollary 6.3 the solution $y(x)$ is of the form (6.5) with $\theta>0$. In the following relations, we use (6.3), (6.5), (3.9), (1.4), (6.8), (1.5) and

$$1=\lim_{x\to-\infty}y(x)d(x)=\lim_{x\to-\infty}\frac{v'(x)+\theta u'(x)}{v(x)+\theta u(x)}d(x)=\lim_{x\to-\infty}\frac{\theta^{-1}\frac{v'(x)}{u'(x)}+1}{\theta^{-1}\frac{v(x)}{u(x)}+1}\,\frac{u'(x)}{u(x)}d(x)$$
$$=\lim_{x\to-\infty}y_1(x)d(x)=-1.$$

Contradiction. Hence $y(x)=y_2(x)$, $x\in R$. The part of assertion A) related to $y_1(x)$ can be proved similarly.

Let us prove B). Note that $y_+(x)=y_1(x)$ if and only if $c_1=0$ in (6.1). In fact, for $x\in[c,\infty)$ we have

$$y_+(x)=y_1(x)\quad\Leftrightarrow\quad\frac{c_1v'(x)+c_2u'(x)}{c_1v(x)+c_2u(x)}=\frac{u'(x)}{u(x)}$$
$$\Leftrightarrow\quad c_1(v'(x)u(x)-u'(x)v(x))=0\quad\Leftrightarrow\quad c_1=0.$$

Thus the condition $y_+(x)\neq y_1(x)$ implies that in this case we have $c_1\neq0$ in (6.1). Hence, as in the proof of A) given above, we obtain

$$\lim_{x\to\infty}y_+(x)d(x)=\lim_{x\to\infty}\frac{c_1v'(x)+c_2u'(x)}{c_1v(x)+c_2u(x)}d(x)$$
$$=\lim_{x\to\infty}\frac{1+\frac{c_2}{c_1}\,\frac{u'(x)}{v'(x)}}{1+\frac{c_2}{c_1}\,\frac{u(x)}{v(x)}}\,\frac{v'(x)}{v(x)}dx=\lim_{x\to\infty}y_2(x)d(x)=1.$$

The converse statement is an obvious consequence of (1.34). Assertion C) can be proved in the same way as assertion B). $\square$

7. Asymptotics of the Otelbaev function at infinity

The problem that is considered in this section arises as a result of attempts to use Theorems 1.4 and 1.5 in order to study concrete equations (1.1) and (1.32). It is easily seen that to study theoretical problems related to asymptotic behaviour of the function $\rho(x)$ at infinity, one can use formula (1.19) without additional restrictions to $q(x)$. (See, for example, Theorem 1.8. Another such example was given in [3] where (1.17) helped to find

asymptotics at infinity of the distribution function of the spectrum of the Sturm-Liouville operator.) However, to apply Theorems 1.4 and 1.5 to concrete equations, one has to know the asymptotic estimates of $d(x)$ for $|x| \to \infty$. The proof of such estimates is a separate technical problem which is not at all related to the initial question on the properties of $\rho(x)$ for $|x| \to \infty$. To solve this problem, additional requirements different from the conditions of Theorem 1.4 are imposed on the function $q(x)$. In [3], such a requirement is condition (2.1).

In the following theorem, we find an asymptotics of $d(x)$ at infinity under condition (1.2) and some additional requirements which are more convenient for practical checking than the corresponding conditions from [3].

Theorem 7.1. *Suppose that $0 \le q(x) \in L_1^{\text{loc}}(R)$, $x \in R$ and one can represent the function $q(x)$ in the form*

$$q(x) = q_1(x) + q_2(x), \qquad x \in R, \tag{7.1}$$

where $q_1(x)$ is positive for $x \in R$ and twice differentiable for $|x| \gg 1$ and $q_2(x) \in L_1^{\text{loc}}(R)$. Denote

$$A(x) = [0, 2q_1(x)^{-1/2}], \quad x \in R, \tag{7.2}$$

$$\varkappa_1(x) = \frac{1}{q_1(x)^{3/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_1''(\xi) d\xi \right|, \quad |x| \gg 1, \tag{7.3}$$

$$\varkappa_2(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi) d\xi \right|, \quad x \in R. \tag{7.4}$$

Then if the following condition holds:

$$\varkappa_1(x) \to 0, \qquad \varkappa_2(x) \to 0 \qquad \text{as} \qquad |x| \to \infty, \tag{7.5}$$

condition (1.2) *also holds and for every $x \in R$, the equation* (1.11) *has a unique positive solution $d(x)$. Moreover,*

$$d(x) = \frac{1 + \delta(x)}{\sqrt{q_1(x)}}, \quad |\delta(x)| \le 2(\varkappa_1(x) + \varkappa_2(x)) \quad \text{for} \quad |x| \gg 1, \tag{7.6}$$

$$c^{-1} \le d(x)\sqrt{q_1(x)} \le c \qquad \text{for} \qquad x \in R. \tag{7.7}$$

Proof of Theorem 7.1.

We need the following lemma.

Lemma 7.2. *Suppose that q_1 satisfies the hypothesis of Theorem 7.1. For a given $x \in R$, consider the following equation in $\hat d \ge 0$:*

$$S(\hat d) = 2, \qquad S(\hat d) = \hat d \int_{x-\hat d}^{x+\hat d} q_1(t)dt. \tag{7.8}$$

Equation (7.8) has a unique positive solution $\hat d(x)$; moreover (see (7.3))

$$\hat d(x) = \frac{1+\delta_1(x)}{\sqrt{q_1(x)}}, \qquad |\delta_1(x)| \le \varkappa_1(x) \qquad \text{for} \qquad |x| \gg 1, \tag{7.9}$$

$$c^{-1} \le \hat d(x)\sqrt{q_1(x)} \le c \qquad \text{for} \qquad x \in R. \tag{7.10}$$

Proof. Clearly, $S(0) = 0$, $S(\hat d) \to \infty$ as $\hat d \to \infty$ and $S(\hat d)$ is monotone increasing in $\hat d \ge 0$. This implies that for every $x \in R$ equation (7.8) has a unique positive solution $\hat d(x)$. To estimate $\hat d(x)$, let us write the function $S(\hat d)$ in the form (7.11):

$$\begin{aligned} S(\hat d) &= \hat d \int_{x-\hat d}^{x+\hat d} q_1(t)dt = \hat d \int_0^{\hat d} [q_1(x+t) + q_1(x-t)]dt \\ &= 2q_1(x)\hat d^2 + \hat d \int_0^{\hat d} [q_1(x+t) - 2q_1(x) + q_1(x-t)]dt \\ &= 2q_1(x)\hat d^2 + \hat d \int_0^{\hat d} \int_0^t \int_{x-\xi}^{x+\xi} q_1''(s)ds d\xi dt. \end{aligned} \tag{7.11}$$

Set (see (7.3))

$$\eta(x) = (1+\varkappa_1(x))q_1(x)^{-1/2}, \qquad |x| \gg 1.$$

By (7.5), $\varkappa_1(x) \le 1$ for all $|x| \gg 1$, and therefore $\eta(x) \in A(x)$ for all $|x| \gg 1$ (see (7.2)). Then from (7.11) and (7.3), it follows that

$$\begin{aligned} S(\eta(x)) &= 2(1+\varkappa_1(x))^2 + \frac{1+\varkappa_1(x)}{\sqrt{q_1(x)}} \int_0^{\eta(x)} \int_0^t \int_{x-\xi}^{x+\xi} q_1''(s)ds d\xi dt \\ &\ge 2(1+\varkappa_1(x))^2 - \frac{1}{2}\frac{(1+\varkappa_1(x))^2}{q_1(x)^{3/2}} \sup_{\xi \in A(x)} \left| \int_{x-\xi}^{x+\xi} q_1''(s)ds \right| \\ &= 2(1+\varkappa_1(x))^2 - \frac{\varkappa_1(x)(1+\varkappa_1(x))^3}{2} \ge 2 + 2\varkappa_1(x) \ge 2. \end{aligned} \tag{7.12}$$

By Lemma 2.2, (7.12) implies the inequality

$$\hat d(x) \le \eta(x) = (1+\varkappa_1(x))q_1(x)^{-1/2} \qquad \text{for all} \qquad |x| \gg 1. \tag{7.13}$$

Let now

$$\eta(x) = (1+\varkappa_1(x))^{-1}q_1(x)^{-1/2}, \qquad |x| \gg 1.$$

Then, as above, we have $\eta(x)\in A(x)$ and using (7.11) and (7.3), we obtain

$$\begin{aligned} S(\eta(x)) &= \frac{2}{(1+\varkappa_1(x)^2}+\frac{1}{1+\varkappa_1(x)}\frac{1}{\sqrt{q_1(x)}}\int_0^{\eta(x)}\int_0^t\int_{x-\xi}^{x+\xi}q_1''(s)dsdtd\xi \\ &\le \frac{2}{1+\varkappa_1(x))^2}+\frac{1}{2}\frac{\eta(x)^2}{1+\varkappa_1(x)}\frac{1}{\sqrt{q_1(x)}}\sup_{\xi\in A(x)}\left|\int_{x-\xi}^{x+\xi}q_1''(s)ds\right| \\ &\le \frac{2}{(1+\varkappa_1(x))^2}+\frac{\varkappa_1(x)}{2(1+\varkappa_1(x))^3}\le 2. \end{aligned} \tag{7.14}$$

By Lemma 2.2, (7.14) implies the inequality

$$\hat d(x)\ge \eta(x)=(1+\varkappa_1(x)^{-1}q_1(x)^{-1/2}\qquad \text{for all}\qquad |x|\gg 1. \tag{7.15}$$

Estimates (7.13) and (7.15) yield (7.9). Inequalities (7.10) follows from (7.9) and Lemma 2.6. □

We now prove Theorem 7.1. Consider the following equation in $d\ge 0:$

$$S(d)=2,\qquad S(d)=d\int_{x-d}^{x+d}q(\xi)d\xi. \tag{7.16}$$

Let $\eta(x)=(1+\varkappa_2(x)\hat d(x)$ and $|x|\gg 1$ (see (7.4)). From (7.5) and (7.9), it follows that $\eta(x)\in A(x)$ for all $|x|\gg 1$. For such an x, Lemma 7.2 and Lemma 7.4 imply

$$\begin{aligned} S(\eta(x)) &= \eta(x)\int_{x-\eta(x)}^{x+\eta(x)}q(t)dt=\eta(x)\int_{x-\eta(x)}^{x+\eta(x)}q_1(t)dt+\eta(x)\int_{x-\eta(x)}^{x+\eta(x)}q_2(t)dt \\ &\ge (1+\varkappa_2(x))\hat d(x)\int_{x-\hat d(x)}^{x+\hat d(x)}q_1(t)dt+\eta(x)\int_{x-\eta(x)}^{x+\eta(x)}q_2(t)dt \\ &\ge 2(1+\varkappa_2(x))-\frac{(1+\varkappa_1(x))(1+\varkappa_2(x))}{\sqrt{q_1(x)}}\left|\int_{x-\eta(x)}^{x+\eta(x)}q_2(t)dt\right| \\ &\ge 2(1+\varkappa_2(x))-\varkappa_2(x)(1+\varkappa_2(x))(1+\varkappa_1(x))\ge 2. \end{aligned} \tag{7.17}$$

From (7.17) and the definition (7.16) of the function $S(d)$, it is not hard to conclude (see §2) that for all $|x|\gg 1$ equation (7.16) has a unique positive root $d(x)$. This implies that this property of equation (7.16) remains true for all $x\in R$ and therefore, in particular, (1.2) holds. Furthermore, (7.17) and Lemma 2.2 lead to the inequality

$$d(x)\le\eta(x)=(1+\varkappa_2(x))\hat d(x)\qquad\text{for all}\qquad |x|\gg 1. \tag{7.18}$$

Set

$$\eta(x)=(1+\varkappa_2(x))^{-1}\hat d(x)\qquad\text{for}\qquad |x|\gg 1.$$

From (7.5) and (7.9), it follows that $\eta(x) \in A(x)$ for all $|x| \gg 1$. Then according to (7.4), Lemma 7.2 implies

$$\begin{aligned}
S(\eta(x)) &= \eta(x)\int_{x-\eta(x)}^{x+\eta(x)} q(t)dt = \eta(x)\int_{x-\eta(x)}^{x+\eta(x)} q_1(t)dt + \eta(x)\int_{x-\eta(x)}^{x+\eta(x)} q_2(t)dt \\
&\le \frac{1}{1+\varkappa_2(x)}\hat d(x)\int_{x-\hat d(x)}^{x+\hat d(x)} q_1(t)dt + \eta(x)\int_{x-\eta(x)}^{x+\eta(x)} q_2(t)dt \\
&\le \frac{2}{1+\varkappa_2(x)} + \frac{1+\varkappa_1(x)}{1+\varkappa_2(x)}\,\frac{1}{\sqrt{q_1(x)}}\sup_{t\in A(x)}\left|\int_{x-t}^{x+t} q_2(\xi)d\xi\right| \\
&= \frac{2}{1+\varkappa_2(x)} + \varkappa_2(x)\frac{1+\varkappa_1(x)}{1+\varkappa_2(x)} \le 2. \qquad (7.19)
\end{aligned}$$

Hence by Lemma 2.2 and estimate (7.19), we have

$$d(x) \ge \eta(x) = (1+\varkappa_2(x))^{-1}\hat d(x) \qquad \text{for all} \qquad |x| \gg 1. \qquad (7.20)$$

Set

$$d(x) = (1+\alpha(x))\hat d(x), \qquad |x| \gg 1.$$

Then using the facts proved above, we obtain $|\alpha(x)| \le \varkappa_2(x)$ for all $|x| \gg 1$. Therefore, taking into account (7.9), for all $|x| \gg 1$, we get

$$d(x) = (1+\alpha(x))\hat d(x) = \frac{(1+\alpha(x))(1+\delta_1(x))}{\sqrt{q_1(x)}} := \frac{1+\delta(x)}{\sqrt{q_1(x)}}$$

$$\Rightarrow\ |\delta(x)| \le |\alpha(x)| + |\delta_1(x)| + |\alpha(x)\delta_1(x)| \le 2(|\alpha(x)| + |\delta_1(x)|) \le 2(\varkappa_1(x)+\varkappa_2(x)) \ \Rightarrow\ (7.6).$$

Inequalities (7.7) follows from (7.6) and Lemma 2.6. $\square$

8. Example

In this section, we consider equation (1.1) and (1.33) where

$$q(x) = \begin{cases} 1, & \text{if} \quad |x| \le 1 \\ |x|^\alpha + |x|^\alpha \cos|x|^\beta, & \text{if} \quad |x| > 1 \end{cases} \qquad (8.1)$$

under the conditions

$$\alpha > -2, \qquad \beta > 1 + \frac{\alpha}{2}. \qquad (8.2)$$

Our goal is to use Theorems 1.4, 1.8 and 7.1 for finding their analogues in the particular case (8.1). For the reader's convenience, we present the statements proved below as separate theorems although these "theorems" are, of course, just examples to the statements proved above.

Theorem 8.1. *Suppose that $q(x)$ is of the form* (8.1). *Then for every $x \in R$, equation* (1.11) *has a unique solution $d(x)$. If, in addition, condition* (8.2) *holds, then for all $|x| \gg 1$, we have*

$$d(x) = \frac{1+\delta(x)}{|x|^{\alpha/2}}, \qquad |\delta(x)| \le \frac{c}{|x|^{\gamma}}, \tag{8.3}$$

where $\gamma = \min\left\{2, \beta - 1 - \frac{\alpha}{2}\right\}$.

Remark 8.2. Since the function $q(x)$ in (8.1) is even, throughout the sequel we will assume $x \ge 0$. Final results will be written for all $x \in R$.

Proof of Theorem 8.1. In the case (8.1), relations (1.2) easily follows from the shape of the graph of $q(x)$. Then by Lemma 2.1, for every $x \in R$ there exists a unique positive solution $d(x)$ of equation (1.11). To prove formula (8.3), we apply Theorem 7.1. Let $x \gg 1$, $q_1(x) = x^\alpha$, $q_2(x) = x^\alpha \cos x^\beta$. Then (see (7.2))

$$A(x) = [0, 2q_1(x)^{-1/2}] = [0, 2x^{-\alpha/2}]. \tag{8.4}$$

From (8.2) for $t \in A(x)$ and $\xi \in [x-t, x+t]$, we get the inequalities

$$|\xi| \le x + t \le x + 2x^{-\alpha/2} = x\left(1 + 2x^{-1-\frac{\alpha}{2}}\right) \le 3x \quad \text{for} \quad x \gg 1, \tag{8.5}$$

$$|\xi| \ge x + t \ge x - 2x^{-\alpha/2} = x\left(1 - 2x^{-1-\frac{\alpha}{2}}\right) \ge 3^{-1}x \quad \text{for} \quad x \gg 1. \tag{8.6}$$

Inequalities (8.5)–(8.6) are used below to estimate $\varkappa_1(x)$ for $x \gg 1$ (see (7.3)):

$$\begin{aligned}
\varkappa_1(x) &= \frac{1}{q_1(x)^{3/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_1''(\xi) d\xi \right| = \frac{1}{x^{3\alpha/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \alpha(\alpha-1)\xi^{\alpha-2} d\xi \right| \\
&\le \frac{c}{x^{3\alpha/2}} x^{\alpha-2} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} 1 d\xi \right| = \frac{c(\alpha)}{x^2}.
\end{aligned} \tag{8.7}$$

Denote $a(x) = \left[x - 2x^{\alpha/2}, x + 2x^{\alpha/2}\right]$ for $x \gg 1$. In the following estimate for $\varkappa_2(x)$ (see (7.4)) for $x \gg 1$, we use relations (8.5)–(8.6) and the second mean theorem ([11, Ch.12, §12, no.3]):

$$\begin{aligned}
\varkappa_2(x) &= \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi) d\xi \right| = \frac{1}{x^{\alpha/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \xi^{\alpha-\beta+1} \frac{(\beta\xi^{\beta-1}\cos\xi^\beta) d\xi}{\beta} \right| \\
&\le c \frac{x^{\alpha-\beta+1}}{x^{\alpha/2}} \sup_{S_1, S_2 \in a(x)} \left| \int_{S_1}^{S_2} \beta\xi^{\beta-1} \cos\xi^\beta d\xi \right| \le \frac{c}{x^{\beta-1-\alpha/2}}.
\end{aligned} \tag{8.8}$$

From (8.2), (8.8) and (8.7), we get condition (7.5). Now (8.3) follows from Theorem 7.1. □

Theorem 8.3. *Suppose that $q(x)$ is of the form* (8.1) *and conditions* (8.2) *hold. Then $q(x) \in H$.*

Proof. Since in this case condition (1.2) holds (see the proof of Theorem 8.1), it remains to find a function $k(x)$ satisfying the requirements of Definition 1.2. Let m be a positive number which will be chosen later. Set

$$k(x)=\begin{cases}2, & \text{if} \quad |x|\le 2^{\frac{m}{\alpha+2}}\\ |x|^{\frac{\alpha+2}{m}}, & \text{if} \quad |x|\ge 2^{\frac{m}{\alpha+2}}\end{cases} \tag{8.9}$$

From (8.9) and (8.2), it follows that (1.12) holds. Let us check (1.13). Let $x\gg 1$ and $m>2$. Then from (8.3) and (8.9) it follows that

$$\frac{k(x)d(x)}{x}\le c\frac{x^{\frac{\alpha+2}{m}}}{x^{1+\frac{\alpha}{2}}}=cx^{\frac{(\alpha+2)(2-m)}{2m}}\to 0 \qquad \text{as} \qquad x\to\infty. \tag{8.10}$$

Therefore from (8.10) for $t\in[x-k(x)d(x),x+k(x)d(x)]$ and $x\gg 1$, we get

$$\begin{aligned} t&\le x+k(x)d(x)=x\left[1+\frac{k(x)d(x)}{x}\right]\le 2x\\ t&\ge x-k(x)d(x)=x\left[1-\frac{k(x)d(x)}{x}\right]\ge \frac{x}{2}.\end{aligned} \tag{8.11}$$

Inequalities (1.13) for $x\gg 1$ and $t\in[x-k(x)d(x),x+k(x)d(x)]$ follow from (8.11):

$$\frac{k(t)}{k(x)}=\left(\frac{t}{x}\right)^{\frac{\alpha+2}{m}}\le 2^{\frac{\alpha+2}{m}}, \qquad \frac{k(t)}{k(x)}=\left(\frac{t}{x}\right)^{\frac{\alpha+2}{m}}\ge\left(\frac{1}{2}\right)^{\frac{\alpha+2}{m}}. \tag{8.12}$$

Estimates (1.13) for all $x\in R$ can now be derived from (8.12) taking into account that the functions under consideration are even and using Lemma 2.6. Let us check (1.14). It is easy to see that in order to estimate $\Phi(x)$ (see (1.14)), one can use estimates for $\Phi_1(x)$, $\Phi_2(x)$ and $\Phi_3(x)$:

$$\begin{aligned}\Phi(x)&=k(x)d(x)\sup_{z\in[0,k(x)d(x)]}\left|\int_0^z[q(x+t)-q(x-t)]dt\right|\\ &\le k(x)d(x)\sup_{z\in[0,k(x)d(x)]}\left|\int_0^z[q_1(x+t)-q_1(x-t)]dt\right|\\ &\quad+k(x)d(x)\sup_{z\in[0,k(x)d(x)]}\left|\int_x^{x+z}q_2(\xi)d\xi\right|\\ &\quad+k(x)d(x)\sup_{z\in[0,k(x)d(x)]}\left|\int_{x-z}^{x}q_2(\xi)d\xi\right|\\ &:=\Phi_1(x)+\Phi_2(x)+\Phi_3(x), \qquad x\in R.\end{aligned} \tag{8.13}$$

Let $m > 6$. Below in the estimate of $\Phi_1(x)$ for $x \gg 1$, we use relations (8.11), (8.12), (8.9) and (8.3):

$$\begin{aligned}\Phi_1(x) &= k(x)d(x) \sup_{z\in[0,k(x)d(x)]}\left|\int_0^z\int_{x-t}^{x+t} q_1'(\xi)dt\right| = k(x)d(x)\sup_{z\in[0,k(x)d(x)]}\left|\int_0^z\int_{x-t}^{x+t}\alpha\xi^{\alpha-1}d\xi dt\right| \\ &\le ck(x)d(x)x^{\alpha-1}\sup_{z\in[0,k(x)d(x)]}\left|\int_0^z\int_{x-t}^{x+t}d\xi dt\right| = c(k(x)d(x))^3x^{\alpha-1} \\ &\le c\frac{x^{\frac{3(\alpha+2)}{m}}}{x^{\frac{3\alpha}{2}}}\cdot x^{\alpha-1} = cx^{\frac{(\alpha+2)(6-m)}{2m}} \le c. \qquad (8.14)\end{aligned}$$

Since the functions under consideration are even and $\Phi_1(x)$ is continuous for $x \in R$, it is not hard to prove that inequalities (7.14) (perhaps with a bigger constant c) hold for all $x \in R$. Let us now consider $\Phi_2(x)$ and $\Phi_3(x)$. We shall prove that these functions are bounded for $x \in R$ and $m \gg 1$; since the proof is the same for both functions, below we only estimate $\Phi_2(x)$. From (8.2) it follows that there exists m_0 such that for all $m \ge m_0 \ge 7$, the following inequalities hold:

$$\beta \ge (\alpha+2)\left(\frac{1}{2}+\frac{1}{m}\right) > \frac{\alpha+2}{2} = 1+\frac{\alpha}{2} \qquad (8.15)$$

$$\Rightarrow \qquad m_0 \overset{\text{def}}{=} \min_{m\ge 7}\left\{m : \frac{(\alpha+2)(m+2)}{2m} \le \beta\right\}. \qquad (8.16)$$

Denote $b(x) = [x, x+k(x)d(x)]$. Below for $x \gg 1$, we use relations (8.11), (8.3), the second mean theorem [5, Ch.11, §2, no.3], (8.15) and (8.16):

$$\begin{aligned}\Phi_2(x) &= k(x)d(x)\sup_{t\in[0,k(x)d(x)]}\left|\int_{x-t}^{x+t}\xi^{\alpha-\beta+1}\frac{[\beta\xi^{\beta-1}\cos\xi^\beta]}{\beta}d\xi\right| \\ &\le ck(x)d(x)x^{\alpha-\beta+1}\sup_{s_1,s_2\in[0,k(x)d(x)]}\left|\int_{s_1}^{s_2}\beta\xi^{\beta-1}\cos\xi^\beta d\xi\right| \\ &\le ck(x)d(x)x^{\alpha-\beta+1} \le c\frac{x^{\frac{\alpha+2}{m_0}}}{x^{\frac{\alpha}{2}}}x^{\alpha-\beta+1} = cx^{\frac{(\alpha+2)(m_0+2)}{2m_0}-\beta} \le c. \qquad (8.17)\end{aligned}$$

As in the case of $\Phi_1(x)$ above, one can extend estimate (8.17) to the whole axis (perhaps with a bigger constant c). The statement of the theorem now follows from (8.13). ☐

Corollary 8.4. *Suppose that the function $q(x)$ is defined by equality* (8.1) *under condition* (8.2) *and $\rho(x)$ is defined by equalities* (1.8). *Then for all $|x| \gg 1$, we have the following asymptotic formula:*

$$\rho(x) = \frac{1+\varepsilon(x)}{2|x|^{\alpha/2}}, \qquad |\varepsilon(x)| \le \frac{c}{|x|^{\gamma_0}}. \qquad (8.18)$$

Here $\gamma_0 = \min\left\{2, \beta-\frac{\alpha}{2}-1, \frac{\alpha+2}{2m_0}\right\}$ and m_0 is the number from (8.16).

Proof. Formula (8.18) follows from Theorems 8.3, 8.1, 1.4 and the final choice of $k(x)$ made in the course of the proof of Theorem 8.3 and (1.20). □

Corollary 8.5. *Consider the Riccati equation* (1.33) *in the case* (8.1) *under condition* (8.2). *The following assertions hold for this equation:*

A) *There exists a unique solution* $y_1(x)$ $(y_2(x))$ *of equation* (1.32) *defined for all* $x \in R$ *and satisfying the equalities*

$$\lim_{x\to-\infty} y_1(x)|x|^{-\alpha/2} = \lim_{x\to\infty} y_1(x)x^{-\alpha/2} = -1$$
$$\left(\lim_{x\to-\infty} y_2(x)|x|^{-\alpha/2} = \lim_{x\to\infty} y_2(x)x^{-\alpha/2} = 1\right).$$

B) *Let* $y_+(x)$ *be a solution of* (1.32) *defined on* $[c,\infty)$ *for some* $c \in R$. *Then* $y_+(x) \neq y_1(x)$ *for* $x \in [c,\infty)$ *if and only if*

$$\lim_{x\to\infty} y_+(x)x^{-\alpha/2} = 1.$$

C) *Let* $y_-(x)$ *be a solution of* (1.32) *defined on* $(-\infty, c]$ *for some* $c \in R$. *Then* $y_-(x) \neq y_2(x)$ *for* $x \in (-\infty, c]$ *if and only if*

$$\lim_{x\to-\infty} y_-(x)|x|^{-\alpha/2} = -1.$$

Proof. This is a consequence of Theorems 8.3, 8.1 and 1.8. □

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